Bayesian Time-Series Econometrics

Book 2 - algebraic derivations

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First edition

Bayesian Time-Series Econometrics

Cover illustration: Thomas Bayes (d. 1761) in Terence O'Donnell, History of Life Insurance in Its Formative Years (Chicago: American Conservation Co:, 1936), p. 335.

To my wife, Mélanie.

To my sons, Tristan and Arnaud.

Contents

PART I

Bayesian statistics

CHAPTER 3

Three applied examples

derivations for equation (1.3.3)

$$f(y|p) = \prod_{i=1}^{n} p^{y_i}(1-p)^{1-y_i} = p^{\sum_{i=1}^{n} y_i}(1-p)^{\sum_{i=1}^{n} 1-y_i} = p^m(1-p)^{n-m} \tag{a.1.3.1}$$

derivations for equation (1.3.5)

The derivative is given by:

$$\frac{dlog(f(y|p))}{dp} = \frac{m}{p} - \frac{n-m}{1-p} \tag{a.1.3.2}$$

Set the value to 0 and solve for p:

$$\begin{aligned}
&\frac{m}{p} - \frac{n-m}{1-p} = 0 \\
\Leftrightarrow\ &\frac{m}{p} = \frac{n-m}{1-p} \\
\Leftrightarrow\ &m(1-p) = p(n-m) \\
\Leftrightarrow\ &m - mp = np - mp \\
\Leftrightarrow\ &m = np \\
\Leftrightarrow\ &p = \frac{m}{n}
\end{aligned} \tag{a.1.3.3}$$

derivations for equation (1.3.11)

$$f(y|p) = \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} = \frac{\prod_{i=1}^{n} \lambda^{y_i} \prod_{i=1}^{n} e^{-\lambda}}{\prod_{i=1}^{n} y_i!} = \frac{\lambda^{\sum_{i=1}^{n} y_i} e^{-n\lambda}}{\prod_{i=1}^{n} y_i!} \tag{a.1.3.4}$$

derivations for equation (1.3.13)

The derivative is given by:

$$\frac{dlog(f(y|\lambda))}{d\lambda} = \frac{\sum_{i=1}^{n} y_i}{\lambda} - n \tag{a.1.3.5}$$

Set the value to 0 and solve for λ:

$$\begin{aligned}
&\frac{\sum_{i=1}^{n} y_i}{\lambda} - n = 0 \\
\Leftrightarrow\ &\frac{\sum_{i=1}^{n} y_i}{\lambda} = n \\
\Leftrightarrow\ &\lambda = \frac{1}{n}\sum_{i=1}^{n} y_i
\end{aligned} \tag{a.1.3.6}$$

derivations for equation (1.3.16)

$$\begin{aligned}
& \lambda^{\sum_{i=1}^n y_i}\ e^{-n\lambda} \times \lambda^{a-1}\ e^{-\lambda/b} \\
=\ & \lambda^{a+\sum_{i=1}^n y_i - 1}\ e^{-\lambda(n+1/b)} \\
=\ & \lambda^{a+\sum_{i=1}^n y_i - 1}\ e^{-\lambda/(n+1/b)^{-1}}
\end{aligned} \tag{a.1.3.7}$$

Now:

$$(n+1/b)^{-1} = \frac{1}{n+1/b} = \frac{b}{bn+1} \tag{a.1.3.8}$$

Hence:

$$\begin{aligned}
& \lambda^{\sum_{i=1}^n y_i}\ e^{-n\lambda} \times \lambda^{a-1}\ e^{-\lambda/b} \\
=\ & \lambda^{a+\sum_{i=1}^n y_i - 1}\ e^{-\lambda/\frac{b}{bn+1}}
\end{aligned} \tag{a.1.3.9}$$

derivations for equation (1.3.19)

$$\begin{aligned}
& f(y|\mu) \\
=\ & \prod_{i=1}^n (2\pi\sigma)^{-1/2} exp\left(-\frac{1}{2}\frac{(y_i-\mu)^2}{\sigma}\right) \\
=\ & \prod_{i=1}^n (2\pi\sigma)^{-1/2}\ \prod_{i=1}^n exp\left(-\frac{1}{2}\frac{(y_i-\mu)^2}{\sigma}\right) \\
=\ & (2\pi\sigma)^{-n/2}\ exp\left(-\frac{1}{2}\sum_{i=1}^n \frac{(y_i-\mu)^2}{\sigma}\right)
\end{aligned} \tag{a.1.3.10}$$

derivations for equation (1.3.21)

The derivative is given by:

$$\frac{dlog(f(y|\mu))}{d\mu} = \sum_{i=1}^n \frac{(y_i-\mu)}{\sigma} \tag{a.1.3.11}$$

Set the value to 0 and solve for μ:

$$\begin{aligned}
& \sum_{i=1}^n \frac{(y_i-\mu)}{\sigma} = 0 \\
\Leftrightarrow\ & \sum_{i=1}^n (y_i-\mu) = 0 \\
\Leftrightarrow\ & \sum_{i=1}^n y_i - n\mu = 0 \\
\Leftrightarrow\ & \sum_{i=1}^n y_i = n\mu \\
\Leftrightarrow\ & \mu = \frac{1}{n}\sum_{i=1}^n y_i
\end{aligned} \tag{a.1.3.12}$$

derivations for equation (1.3.24)

First group the exponential terms:

$$\pi(\mu|y) \propto exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}\right) \times exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right) = exp\left(-\frac{1}{2}\left[\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}+\frac{(\mu-m)^2}{v}\right]\right) \tag{a.1.3.13}$$

Develop the term within the square bracket:

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}+\frac{(\mu-m)^2}{v} \\
= \; &\frac{1}{\sigma}\sum_{i=1}^{n}(y_i^2+\mu^2-2\mu y_i)+\frac{1}{v}(\mu^2+m^2-2\mu m) \\
= \; &\frac{1}{\sigma}\left(\sum_{i=1}^{n}y_i^2+n\mu^2-2\mu\sum_{i=1}^{n}y_i)\right)+\frac{1}{v}(\mu^2+m^2-2\mu m)
\end{aligned} \tag{a.1.3.14}$$

Group the terms:

$$= \; \mu^2\left(\frac{n}{\sigma}+\frac{1}{v}\right)-2\mu\left(\frac{1}{\sigma}\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\frac{1}{\sigma}\sum_{i=1}^{n}y_i^2+\frac{m^2}{v} \tag{a.1.3.15}$$

Set back in (a.1.3.13):

$$\pi(\mu|y) \propto exp\left(-\frac{1}{2}\left[\mu^2\left(\frac{n}{\sigma}+\frac{1}{v}\right)-2\mu\left(\frac{1}{\sigma}\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\frac{1}{\sigma}\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}\right]\right) \tag{a.1.3.16}$$

CHAPTER 4

Further aspects of Bayesian priors and posteriors

derivations for equation (1.4.5)

First group the terms:

$$
\begin{aligned}
&\pi(\mu,\sigma|y) \\
\propto\ & \sigma^{-n/2}\, exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}\right)\times\sigma^{-1/2}exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v\sigma}\right)\times\sigma^{-\alpha/2-1}exp\left(-\frac{\delta}{2\sigma}\right) \\
=\ & \sigma^{-(n+\alpha)/2-1}\times\sigma^{-1/2}\times exp\left(-\frac{1}{2\sigma}\left[\sum_{i=1}^{n}(y_i-\mu)^2+\frac{(\mu-m)^2}{v}+\delta\right]\right) \qquad \text{(a.1.4.1)}
\end{aligned}
$$

Develop the term in the square bracket:

$$
\begin{aligned}
&\sum_{i=1}^{n}(y_i-\mu)^2+\frac{(\mu-m)^2}{v}+\delta \\
=\ & \sum_{i=1}^{n}(y_i^2+\mu^2-2\mu y_i)+\frac{\mu^2}{v}+\frac{m^2}{v}-2\mu\frac{m}{v}+\delta \\
=\ & \sum_{i=1}^{n}y_i^2+n\mu^2-2\mu\sum_{i=1}^{n}y_i+\frac{\mu^2}{v}+\frac{m^2}{v}-2\mu\frac{m}{v}+\delta \\
=\ & \mu^2\left(n+\frac{1}{v}\right)-2\mu\left(\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}+\delta \qquad \text{(a.1.4.2)}
\end{aligned}
$$

Complete the squares:

$$
= \mu^2\left(n+\frac{1}{v}\right)-2\mu\frac{\bar{v}}{\bar{v}}\left(\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}+\delta+\frac{\bar{m}^2}{\bar{v}}-\frac{\bar{m}^2}{\bar{v}} \qquad \text{(a.1.4.3)}
$$

Define:

$$
\bar{v}=\left(n+\frac{1}{v}\right)^{-1} \qquad \bar{m}=\bar{v}\left(\sum_{i=1}^{n}y_i+\frac{m}{v}\right) \qquad \text{(a.1.4.4)}
$$

Then (a.1.4.3) rewrites:

$$
\begin{aligned}
=\ & \frac{\mu^2}{\bar{v}}+\frac{\bar{m}^2}{\bar{v}}-2\mu\frac{\bar{m}}{\bar{v}}+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}+\delta-\frac{\bar{m}^2}{\bar{v}} \\
=\ & \frac{(\mu-\bar{m})^2}{\bar{v}}+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}+\delta-\frac{\bar{m}^2}{\bar{v}} \qquad \text{(a.1.4.5)}
\end{aligned}
$$

Substituting back (a.1.4.5) in (a.1.4.1) eventually yields:

$$\begin{aligned}
&\pi(\mu,\sigma|y) \\
\propto\ & \sigma^{-(n+\alpha)/2-1} \times \sigma^{-1/2} \times exp\left(-\frac{1}{2\sigma}\left[\frac{(\mu-\bar{m})^2}{\bar{v}} + \sum_{i=1}^{n} y_i^2 + \frac{m^2}{v} + \delta - \frac{\bar{m}^2}{\bar{v}}\right]\right) \\
=\ & \sigma^{-(n+\alpha)/2-1} \times \sigma^{-1/2} \times exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\sigma\bar{v}}\right) \times exp\left(-\frac{1}{2\sigma}\left[\sum_{i=1}^{n} y_i^2 + \frac{m^2}{v} + \delta - \frac{\bar{m}^2}{\bar{v}}\right]\right) \\
=\ & \sigma^{-\bar{\alpha}/2-1} \times \sigma^{-1/2} \times exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\sigma\bar{v}}\right) \times exp\left(-\frac{\bar{\delta}}{2\sigma}\right) \qquad \text{(a.1.4.6)}
\end{aligned}$$

with:

$$\bar{\alpha} = n + \alpha \qquad \bar{\delta} = \sum_{i=1}^{n} y_i^2 + \frac{m^2}{v} + \delta - \frac{\bar{m}^2}{\bar{v}} \qquad \text{(a.1.4.7)}$$

derivations for equation (1.4.10)

Rearrange the terms:

$$\begin{aligned}
&\pi(\mu|y) \\
\propto\ & \Gamma\left(\frac{\bar{\alpha}+1}{2}\right)\left(\frac{\bar{\delta}+(\mu-\bar{m})^2/\bar{v}}{2}\right)^{-\frac{\bar{\alpha}+1}{2}} \\
\propto\ & \left(\frac{\bar{\delta}+(\mu-\bar{m})^2/\bar{v}}{2}\right)^{-\frac{\bar{\alpha}+1}{2}} \\
\propto\ & \left(\bar{\delta}+\frac{(\mu-\bar{m})^2}{\bar{v}}\right)^{-\frac{\bar{\alpha}+1}{2}} \\
=\ & \bar{\delta}\left(1+\frac{(\mu-\bar{m})^2}{\bar{\delta}\bar{v}}\right)^{-\frac{\bar{\alpha}+1}{2}} \\
\propto\ & \left(1+\frac{(\mu-\bar{m})^2}{\bar{\delta}\bar{v}}\right)^{-\frac{\bar{\alpha}+1}{2}} \\
=\ & \left(1+\frac{1}{\bar{\alpha}}\frac{(\mu-\bar{m})^2}{\bar{\delta}\bar{v}/\bar{\alpha}}\right)^{-\frac{\bar{\alpha}+1}{2}} \qquad \text{(a.1.4.8)}
\end{aligned}$$

derivations for equation (1.4.13)

Solve for the derivative:

$$\begin{aligned}
&2\int(\hat{\theta}-\theta)\ \pi(\theta|y)d\theta = 0 \\
\Leftrightarrow\ & \int(\hat{\theta}-\theta)\ \pi(\theta|y)d\theta = 0 \\
\Leftrightarrow\ & \int\hat{\theta}\ \pi(\theta|y)d\theta - \int\theta\ \pi(\theta|y)d\theta = 0 \\
\Leftrightarrow\ & \hat{\theta}\int\pi(\theta|y)d\theta = \int\theta\ \pi(\theta|y)d\theta \\
\Leftrightarrow\ & \hat{\theta} = \int\theta\ \pi(\theta|y)d\theta \qquad \text{(a.1.4.9)}
\end{aligned}$$

derivations for equation (1.4.16)

Rearrange the expression:

$$\begin{aligned} f(y) &= \int\int (2\pi)^{-n/2}\,(2\pi)^{-1/2}\,v^{-1/2}\,\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \\ &\times \sigma^{-n/2} exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}\right) \times \sigma^{-1/2} exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v\sigma}\right) \times \sigma^{-\alpha/2-1} exp\left(-\frac{\delta}{2\sigma}\right) d\mu d\sigma \end{aligned} \qquad \text{(a.1.4.10)}$$

The second row can be recognised as equation (a.1.4.1). Using the same manipulations, one obtains equation (a.1.4.6), and thus the previous expression rewrites as:

$$\begin{aligned} f(y) &= \int\int (2\pi)^{-n/2}\,(2\pi)^{-1/2}\,v^{-1/2}\,\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \\ &\times \sigma^{-1/2} exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\sigma\bar{v}}\right) \times \sigma^{-\bar{\alpha}/2-1} exp\left(-\frac{\bar{\delta}}{2\sigma}\right) d\mu d\sigma \end{aligned} \qquad \text{(a.1.4.11)}$$

with $\bar{m}, \bar{v}, \bar{\alpha}$ and $\bar{\delta}$ defined as in (a.1.4.4) and (a.1.4.7). Now add multiplicative terms to obtain normal and inverse Gamma probability density functions, and take constants out of the integral:

$$\begin{aligned} f(y) &= (2\pi)^{-n/2}\,v^{-1/2}\,\bar{v}^{1/2}\,\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\,\frac{\Gamma(\bar{\alpha}/2)}{\bar{\delta}/2^{\bar{\alpha}/2}} \\ &\times \int\int (2\pi\bar{v}\sigma)^{-1/2} exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\sigma\bar{v}}\right) \times \frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\sigma^{-\bar{\alpha}/2-1} exp\left(-\frac{\bar{\delta}}{2\sigma}\right) d\mu d\sigma \end{aligned} \qquad \text{(a.1.4.12)}$$

The expression can simplify further. Consider only the constant on the first line:

$$\begin{aligned} & (2\pi)^{-n/2}\,v^{-1/2}\,\bar{v}^{1/2}\,\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\,\frac{\Gamma(\bar{\alpha}/2)}{\bar{\delta}/2^{\bar{\alpha}/2}} \\ = &\ 2^{-n/2}\pi^{-n/2}\,v^{-1/2}\,((n+1/v)^{-1})^{1/2}\,\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\,\frac{2^{\bar{\alpha}/2}}{2^{\alpha/2}}\,\frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\ = &\ 2^{-n/2}\pi^{-n/2}\,v^{-1/2}\,(n+1/v)^{-1/2}\,\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\,\frac{2^{(\alpha+n)/2}}{2^{\alpha/2}}\,\frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\ = &\ \pi^{-n/2}\,(1+vn)^{-1/2}\,\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\,\frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \end{aligned} \qquad \text{(a.1.4.13)}$$

Substitute back in (a.1.4.12):

$$\begin{aligned} f(y) &= \pi^{-n/2}\,(1+vn)^{-1/2}\,\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\,\frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\ &\times \int\int (2\pi\bar{v}\sigma)^{-1/2} exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\sigma\bar{v}}\right) \times \frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\sigma^{-\bar{\alpha}/2-1} exp\left(-\frac{\bar{\delta}}{2\sigma}\right) d\mu d\sigma \end{aligned} \qquad \text{(a.1.4.14)}$$

derivations for equation (1.4.19)

Rearrange the expression:

$$\begin{aligned}
&\mathbb{P}(M_i|y) = \frac{f(y|M_i)\ \mathbb{P}(M_i)}{f(y)} \\
\Leftrightarrow\ &\mathbb{P}(M_i|y) = \frac{f(y,M_i)\ /\ \pi(M_i)\ \mathbb{P}(M_i)}{f(y)} \\
\Leftrightarrow\ &\mathbb{P}(M_i|y) = \frac{\int f(y,M_i,\theta_i)\ /\ \pi(M_i) d\theta_i\ \mathbb{P}(M_i)}{f(y)} \\
\Leftrightarrow\ &\mathbb{P}(M_i|y) = \frac{\int \frac{f(y,M_i,\theta_i)}{\pi(M_i,\theta)}\ \frac{\pi(M_i,\theta)}{\pi(M_i)} d\theta_i\ \mathbb{P}(M_i)}{f(y)} \\
\Leftrightarrow\ &\mathbb{P}(M_i|y) = \frac{\int f(y|M_i,\theta_i)\ \pi(\theta|M_i) d\theta_i\ \mathbb{P}(M_i)}{f(y)}
\end{aligned} \tag{a.1.4.15}$$

derivations for equation (1.4.24)

Rearrange the expression to obtain:

$$\begin{aligned}
&f(\hat{y}|y) \\
=\ &\int\int \sigma^{-1/2} exp\left(-\frac{1}{2}\frac{(\hat{y}-\mu)^2}{\sigma}\right) \times \sigma^{-n/2}\ exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}\right) \\
&\quad \times \sigma^{-1/2} exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v\sigma}\right) \times \sigma^{-\alpha/2-1} exp\left(-\frac{\delta}{2\sigma}\right) d\mu d\sigma \\
=\ &\int\int \sigma^{-1/2} exp\left(-\frac{1}{2\sigma}\left[(\hat{y}-\mu)^2 + \sum_{i=1}^{n}(y_i-\mu)^2 + \frac{(\mu-m)^2}{v} + \delta\right]\right) \sigma^{-(\alpha+n+1)/2-1} d\mu d\sigma \\
=\ &\int\int \sigma^{-1/2} exp\left(-\frac{1}{2\sigma}\left[(\hat{y}-\mu)^2 + \sum_{i=1}^{n}(y_i-\mu)^2 + \frac{(\mu-m)^2}{v} + \delta\right]\right) \sigma^{-\hat{\alpha}/2-1} d\mu d\sigma
\end{aligned} \tag{a.1.4.16}$$

with:

$$\hat{\alpha} = \alpha + n + 1 \tag{a.1.4.17}$$

Consider the term in square brackets:

$$
\begin{aligned}
& (\hat{y}-\mu)^2+\sum_{i=1}^{n}(y_i-\mu)^2+\frac{(\mu-m)^2}{v}+\delta \\
= \; & \hat{y}^2+\mu^2-2\hat{y}\mu+\sum_{i=1}^{n}(y_i^2+\mu^2-2y_i\mu)+\frac{\mu^2}{v}+\frac{m^2}{v}-2\mu\frac{m}{v}+\delta \\
= \; & \hat{y}^2+\mu^2-2\hat{y}\mu+\sum_{i=1}^{n}y_i^2+n\mu^2-2\mu\sum_{i=1}^{n}y_i+\frac{\mu^2}{v}+\frac{m^2}{v}-2\mu\frac{m}{v}+\delta \\
= \; & \left(\delta+\hat{y}^2+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}\right)+\mu^2\left(1+n+\frac{1}{v}\right)-2\mu\left(\hat{y}+\sum_{i=1}^{n}y_i+\frac{m}{v}\right) \\
= \; & \left(\delta+\hat{y}^2+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}\right)+\mu^2\left(1+n+\frac{1}{v}\right)-2\mu\frac{\hat{v}}{\hat{v}}\left(\hat{y}+\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\frac{\hat{m}^2}{\hat{v}}-\frac{\hat{m}^2}{\hat{v}} \\
= \; & \left(\delta+\hat{y}^2+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}-\frac{\hat{m}^2}{\hat{v}}\right)+\mu^2\left(1+n+\frac{1}{v}\right)-2\mu\frac{\hat{v}}{\hat{v}}\left(\hat{y}+\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\frac{\hat{m}^2}{\hat{v}}
\end{aligned}
\tag{a.1.4.18}
$$

Define:

$$
\hat{\delta}=\left(\delta+\hat{y}^2+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}-\frac{\hat{m}^2}{\hat{v}}\right) \qquad \hat{v}=\left(1+n+\frac{1}{v}\right)^{-1} \qquad \hat{m}=\hat{v}\left(\hat{y}+\sum_{i=1}^{n}y_i+\frac{m}{v}\right)
\tag{a.1.4.19}
$$

Then (a.1.4.17) becomes:

$$
\begin{aligned}
= \; & \hat{\delta}+\frac{\mu^2}{\hat{v}}-2\mu\frac{\hat{m}}{\hat{v}}+\frac{\hat{m}^2}{\hat{v}} \\
= \; & \hat{\delta}+\frac{(\mu-\hat{m})^2}{\hat{v}}
\end{aligned}
\tag{a.1.4.20}
$$

Substitute back in (a.1.4.16):

$$
\begin{aligned}
& f(\hat{y}|y) \\
= \; & \int\int \sigma^{-1/2}exp\left(-\frac{1}{2\sigma}\left[\hat{\delta}+\frac{(\mu-\hat{m})^2}{\hat{v}}\right]\right)\sigma^{-\hat{\alpha}/2-1}d\mu d\sigma \\
= \; & \int\int \sigma^{-1/2}exp\left(-\frac{1}{2}\frac{(\mu-\hat{m})^2}{\hat{v}\sigma}\right)\times\sigma^{-\hat{\alpha}/2-1}exp\left(-\frac{\hat{\delta}}{2\sigma}\right)d\mu d\sigma \\
= \; & \int \sigma^{-\hat{\alpha}/2-1}exp\left(-\frac{\hat{\delta}}{2\sigma}\right)\int \sigma^{-1/2}exp\left(-\frac{1}{2}\frac{(\mu-\hat{m})^2}{\hat{v}\sigma}\right)d\mu d\sigma
\end{aligned}
\tag{a.1.4.21}
$$

The second integral contains the kernel of a normal distribution with mean $\hat{m}$ and variance $\hat{v}\sigma$. It thus integrates to a constant (not involving $\hat{y}$) and can be relegated to the normalization constant, yielding:

$$
\propto \int \sigma^{-(\hat{\alpha}+1)/2-1}exp\left(-\frac{\hat{\delta}}{2\sigma}\right)d\sigma
\tag{a.1.4.22}
$$

The remaining integral contains the kernel of an inverse Gamma distribution with shape $\hat{\alpha}$ and scale $\hat{\delta}$. It integrates to the reciprocal of the normalization constant of the inverse Gamma distribution (see book1, section 4.3), which does involve $\hat{y}$. The term must thus be retained, yielding:

$$
\begin{aligned}
& f(\hat{y}|y) \\
\propto\ & \Gamma(\hat{\alpha})\,(\hat{\delta}/2)^{-\hat{\alpha}/2} \\
\propto\ & (\hat{\delta}/2)^{-\hat{\alpha}/2} \\
\propto\ & (\hat{\delta})^{-\hat{\alpha}/2} \\
=\ & \left(\delta+\hat{y}^2+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\frac{\hat{m}^2}{\hat{v}}\right)^{-\hat{\alpha}/2} \\
=\ & \left(\delta+\hat{y}^2+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\hat{v}\left[\hat{y}+\sum_{i=1}^{n} y_i+\frac{m}{v}\right]^2\right)^{-\hat{\alpha}/2}
\end{aligned}
\tag{a.1.4.23}
$$

Define:

$$\tilde{m}=\sum_{i=1}^{n} y_i+\frac{m}{v} \tag{a.1.4.24}$$

Then (a.1.4.23) becomes:

$$
\begin{aligned}
=\ & \left(\delta+\hat{y}^2+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\hat{v}[\hat{y}+\tilde{m}]^2\right)^{-\hat{\alpha}/2} \\
=\ & \left(\delta+\hat{y}^2+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\hat{v}\hat{y}^2-\hat{v}\tilde{m}^2-2\hat{v}\tilde{m}\hat{y}\right)^{-\hat{\alpha}/2} \\
=\ & \left(\delta+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\hat{v}\tilde{m}^2+\hat{y}^2(1-\hat{v})-2\hat{v}\tilde{m}\hat{y}\right)^{-\hat{\alpha}/2}
\end{aligned}
\tag{a.1.4.25}
$$

Complete the squares:

$$
\begin{aligned}
=\ & \left(\delta+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\hat{v}\tilde{m}^2+\hat{y}^2(1-\hat{v})-2\hat{v}\frac{\ddot{v}}{\ddot{v}}\tilde{m}\hat{y}+\frac{\ddot{m}^2}{\ddot{v}}-\frac{\ddot{m}^2}{\ddot{v}}\right)^{-\hat{\alpha}/2} \\
=\ & \left(\left[\delta+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\hat{v}\tilde{m}^2-\frac{\ddot{m}^2}{\ddot{v}}\right]+\hat{y}^2(1-\hat{v})-2\hat{v}\frac{\ddot{v}}{\ddot{v}}\tilde{m}\hat{y}+\frac{\ddot{m}^2}{\ddot{v}}\right)^{-\hat{\alpha}/2}
\end{aligned}
\tag{a.1.4.26}
$$

Define:

$$\bar{\alpha}=\alpha+n \qquad \ddot{\delta}=\delta+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\hat{v}\tilde{m}^2-\frac{\ddot{m}^2}{\ddot{v}} \qquad \ddot{v}=(1-\hat{v})^{-1} \qquad \ddot{m}=\hat{v}\ddot{v}\tilde{m} \tag{a.1.4.27}$$

Then (a.1.4.26) becomes:

$$\begin{aligned}
&= \left(\ddot{\delta} + \frac{\hat{y}^2}{\ddot{v}} - 2\hat{y}\frac{\ddot{m}}{\ddot{v}} + \frac{\ddot{m}^2}{\ddot{v}}\right)^{-(\bar{\alpha}+1)/2} \\
&= \left(\ddot{\delta} + \frac{(\hat{y}-\ddot{m})^2}{\ddot{v}}\right)^{-(\bar{\alpha}+1)/2} \\
&= \ddot{\delta}^{-(\bar{\alpha}+1)/2}\left(1 + \frac{(\hat{y}-\ddot{m})^2}{\ddot{\delta}\ddot{v}}\right)^{-(\bar{\alpha}+1)/2} \\
&\propto \left(1 + \frac{(\hat{y}-\ddot{m})^2}{\ddot{\delta}\ddot{v}}\right)^{-(\bar{\alpha}+1)/2} \\
&= \left(1 + \frac{1}{\bar{\alpha}}\,\frac{(\hat{y}-\ddot{m})^2}{\ddot{\delta}\ddot{v}/\bar{\alpha}}\right)^{-(\bar{\alpha}+1)/2} \qquad \text{(a.1.4.28)}
\end{aligned}$$

Finally, reformulate all the messy terms:

$$\begin{aligned}
\ddot{v} &= (1-\hat{v})^{-1} = \frac{1}{1-\hat{v}} = \frac{1}{1-\frac{1}{1+n+\frac{1}{v}}} = \frac{1}{\frac{1+n+\frac{1}{v}-1}{1+n+\frac{1}{v}}} = \frac{1}{\frac{n+\frac{1}{v}}{1+n+\frac{1}{v}}} = \frac{1+n+\frac{1}{v}}{n+\frac{1}{v}} = \frac{v+vn+1}{vn+1} \\
&= 1 + \frac{v}{vn+1} = 1 + \frac{1}{n+1/v} = 1 + \left(n+\frac{1}{v}\right)^{-1} = 1 + \bar{v} \qquad \text{(a.1.4.29)}
\end{aligned}$$

with $\bar{v}$ defined as in (a.1.4.4).

Also:

$$\frac{\hat{v}}{1-\hat{v}} = \frac{\frac{1}{1+n+\frac{1}{v}}}{1-\frac{1}{1+n+\frac{1}{v}}} = \frac{\frac{1}{1+n+\frac{1}{v}}}{\frac{1+n+\frac{1}{v}-1}{1+n+\frac{1}{v}}} = \frac{\frac{1}{1+n+\frac{1}{v}}}{\frac{n+\frac{1}{v}}{1+n+\frac{1}{v}}} = \frac{1}{n+\frac{1}{v}} = \left(n+\frac{1}{v}\right)^{-1} = \bar{v} \qquad \text{(a.1.4.30)}$$

Then:

$$\ddot{m} = \hat{v}\ddot{v}\tilde{m} = \frac{\hat{v}}{1-\hat{v}}\tilde{m} = \bar{v}\tilde{m} = \bar{v}\left(\sum_{i=1}^{n} y_i + \frac{m}{v}\right) = \bar{m} \qquad \text{(a.1.4.31)}$$

with $\bar{m}$ defined as in (a.1.4.4).

Finally:

$$\begin{aligned}
&\hat{v}\tilde{m}^2 + \frac{\ddot{m}^2}{\ddot{v}} = \hat{v}\tilde{m}^2 + (\hat{v}\ddot{v}\tilde{m})^2/\ddot{v} = \hat{v}\tilde{m}^2 + \hat{v}^2\ddot{v}\tilde{m}^2 = \hat{v}\tilde{m}^2(1+\hat{v}\ddot{v}) = \hat{v}\tilde{m}^2\left(1+\frac{\hat{v}}{1-\hat{v}}\right) \\
&= \hat{v}\tilde{m}^2\left(\frac{1-\hat{v}+\hat{v}}{1-\hat{v}}\right) = \hat{v}\tilde{m}^2\left(\frac{1}{1-\hat{v}}\right) = \tilde{m}^2\left(\frac{\hat{v}}{1-\hat{v}}\right) = \tilde{m}^2\bar{v} = \tilde{m}\bar{m} = \frac{\bar{m}}{\bar{v}}\bar{m} = \frac{\bar{m}^2}{\bar{v}} \qquad \text{(a.1.4.32)}
\end{aligned}$$

Substitute in (a.1.4.27) to obtain:

$$\ddot{\delta} = \delta + \sum_{i=1}^{n} y_i^2 + \frac{m^2}{v} - \frac{\bar{m}^2}{\bar{v}} = \bar{\delta} \qquad \text{(a.1.4.33)}$$

with $\bar{\delta}$ defined as in (a.1.4.7).

Substitute (a.1.4.29), (a.1.4.31) and (a.1.4.33) in (a.1.4.28) to eventually obtain:

$$f(\hat{y}|y) \propto \left(1 + \frac{1}{\bar{\alpha}}\,\frac{(\hat{y}-\bar{m})^2}{\bar{\delta}(1+\bar{v})/\bar{\alpha}}\right)^{-(\bar{\alpha}+1)/2} \qquad \text{(a.1.4.34)}$$

CHAPTER 5

Properties of Bayesian estimates

derivations for equation (1.5.1)

The mean of a Beta distribution with shapes a and b is given by $\frac{a}{a+b}$. Given the posterior hyperparameters $\bar{\alpha} = \alpha + m$ and $\bar{\beta} = \beta + n - m$, the posterior mean writes as:

$$
\begin{aligned}
& \mathbb{E}(p|y) \\
= \; & \frac{\bar{\alpha}}{\bar{\alpha} + \bar{\beta}} \\
= \; & \frac{\alpha + m}{\alpha + m + \beta + n - m} \\
= \; & \frac{\alpha + m}{\alpha + \beta + n} \\
= \; & \frac{\alpha}{\alpha + \beta + n} + \frac{m}{\alpha + \beta + n} \\
= \; & \frac{\alpha}{\alpha + \beta} \frac{\alpha + \beta}{\alpha + \beta + n} + \frac{m}{n} \frac{n}{\alpha + \beta + n} \\
= \; & \gamma \; \mathbb{E}(p) + (1 - \gamma) \; \hat{p}
\end{aligned}
\tag{a.1.5.1}
$$

with:

$$
\mathbb{E}(p) = \frac{\alpha}{\alpha + \beta} \qquad \hat{p} = \frac{m}{n} \qquad \gamma = \frac{\alpha + \beta}{\alpha + \beta + n} \tag{a.1.5.2}
$$

derivations for equation (1.5.2)

The mean of a Gamma distribution with shape a and scale b is given by ab. Given the posterior hyperparameters $\bar{a} = a + \sum_{i=1}^{n} y_i$ and $\bar{b} = \frac{b}{bn+1}$, the posterior mean writes as:

$$
\begin{aligned}
& \mathbb{E}(\lambda|y) \\
= \; & \frac{(a + \sum_{i=1}^{n} y_i) b}{bn + 1} \\
= \; & \frac{ab}{bn + 1} + \frac{b \sum_{i=1}^{n} y_i}{bn + 1} \\
= \; & ab \left(\frac{1}{bn + 1} \right) + \frac{\sum_{i=1}^{n} y_i}{n} \left(\frac{bn}{bn + 1} \right) \\
= \; & \gamma \; \mathbb{E}(\lambda) + (1 - \gamma) \; \hat{\lambda}
\end{aligned}
\tag{a.1.5.3}
$$

with:

$$
\mathbb{E}(\lambda) = ab \qquad \hat{\lambda} = \frac{\sum_{i=1}^{n} y_i}{n} \qquad \gamma = \frac{1}{bn + 1} \tag{a.1.5.4}
$$

derivations for equation (1.5.3)

The mean of a normal distribution with mean μ and variance σ is given by μ. Given the posterior hyperparameters $\bar{v} = \left(\frac{n}{\sigma} + \frac{1}{v}\right)^{-1}$ and $\bar{m} = \bar{v}\left(\frac{1}{\sigma}\sum_{i=1}^{n} y_i + \frac{m}{v}\right)$, the posterior variance writes as:

$$\bar{v} = \left(\frac{n}{\sigma} + \frac{1}{v}\right)^{-1} = \frac{1}{n/\sigma + 1/v} = \frac{\sigma}{n + \sigma/v} \tag{a.1.5.5}$$

Then the posterior mean can be expressed as:

$$\begin{aligned}
& \mathbb{E}(\mu|y) \\
= \; & \frac{\sigma}{n+\sigma/v}\left(\frac{1}{\sigma}\sum_{i=1}^{n} y_i + \frac{m}{v}\right) \\
= \; & \frac{1}{n+\sigma/v}\left(\sum_{i=1}^{n} y_i\right) + \frac{\sigma}{n+\sigma/v}\left(\frac{m}{v}\right) \\
= \; & \frac{n}{n+\sigma/v}\left(\frac{\sum_{i=1}^{n} y_i}{n}\right) + \frac{\sigma/v}{n+\sigma/v}\, m \\
= \; & \frac{vn}{vn+\sigma}\left(\frac{\sum_{i=1}^{n} y_i}{n}\right) + \frac{\sigma}{vn+\sigma}\, m \\
= \; & \gamma\, \mathbb{E}(\mu) + (1-\gamma)\, \hat{\mu}
\end{aligned} \tag{a.1.5.6}$$

with:

$$\mathbb{E}(\mu) = m \qquad \hat{\mu} = \frac{\sum_{i=1}^{n} y_i}{n} \qquad \gamma = \frac{\sigma}{vn+\sigma} \tag{a.1.5.7}$$

PART II

Simulation methods

CHAPTER 6

The Gibbs sampling algorithm

derivations for equation (2.6.17)

Combine all the terms to obtain:

$$
\begin{aligned}
& f(y) \\
\approx\ & (2\pi\sigma)^{-n/2}\ \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}\right)\ \frac{(2\pi v)^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi\bar{v})^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)}\ \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\sigma^{-\bar{\alpha}/2-1}\ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
=\ & (2\pi)^{-n/2}\sigma^{-n/2}\ \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}\right)\ \frac{v^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}\bar{v}^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)}\ \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\sigma^{-(\alpha+n)/2-1}\ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
=\ & (2\pi)^{-n/2}\ \exp\ \frac{v^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}\bar{v}^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)}\ \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\exp\left(-\frac{1}{2\sigma}\left[\delta+\sum_{i=1}^{n}(y_i-\mu)^2\right]\right)}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
=\ & (2\pi)^{-n/2}\ \exp\ \frac{v^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}\bar{v}^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)}\ \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}} \\
=\ & 2^{-n/2}\pi^{-n/2}\ \exp\ \frac{v^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}\bar{v}^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)}\ \frac{\frac{\delta^{\alpha/2}}{\Gamma(\alpha/2)}}{\frac{\bar{\delta}^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}}\ \frac{2^{\bar{\alpha}/2}}{2^{\alpha/2}} \\
=\ & \pi^{-n/2}\ \exp\ \frac{v^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}\bar{v}^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)}\ \frac{\frac{\delta^{\alpha/2}}{\Gamma(\alpha/2)}}{\frac{\bar{\delta}^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}} \\
=\ & \pi^{-n/2}\ \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\ \frac{\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}(v/\bar{v})^{1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)}
\end{aligned}
\qquad \text{(a.2.6.1)}
$$

Now:

$$v/\bar{v} = v(n/\sigma+1/v) = vn/\sigma+1 \qquad \text{(a.2.6.2)}$$

Hence:

$$f(y) \approx \pi^{-n/2}\ \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\ \frac{\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}(1+vn/\sigma)^{1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)} \qquad \text{(a.2.6.3)}$$

CHAPTER 7

The Metropolis-Hastings algorithm

derivations for equation (2.7.7)

Rearrange:

$$\begin{aligned}
& \pi(\mu|y,\lambda) \\
\propto\ & \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda)}\right)\times\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right) \\
=\ & \exp\left(-\frac{1}{2}\left[\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda)}+\frac{(\mu-m)^2}{v}\right]\right) \qquad \text{(a.2.7.1)}
\end{aligned}$$

Develop the term within the square bracket:

$$\begin{aligned}
& \sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda)}+\frac{(\mu-m)^2}{v} \\
=\ & \frac{1}{\exp(\lambda)}\sum_{i=1}^{n}(y_i^2+\mu^2-2\mu y_i)+\frac{1}{v}(\mu^2+m^2-2\mu m) \\
=\ & \frac{1}{\exp(\lambda)}\left(\sum_{i=1}^{n}y_i^2+n\mu^2-2\mu\sum_{i=1}^{n}y_i)\right)+\frac{1}{v}(\mu^2+m^2-2\mu m) \qquad \text{(a.2.7.2)}
\end{aligned}$$

Group the terms and complete the squares:

$$\begin{aligned}
=\ & \mu^2\left(\frac{n}{\exp(\lambda)}+\frac{1}{v}\right)-2\mu\left(\frac{1}{\exp(\lambda)}\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\frac{1}{\exp(\lambda)}\sum_{i=1}^{n}y_i^2+\frac{m^2}{v} \\
=\ & \mu^2\left(\frac{n}{\exp(\lambda)}+\frac{1}{v}\right)-2\mu\frac{\bar{v}}{\bar{v}}\left(\frac{1}{\exp(\lambda)}\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\frac{1}{\exp(\lambda)}\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}+\frac{\bar{m}^2}{\bar{v}}-\frac{\bar{m}^2}{\bar{v}} \qquad \text{(a.2.7.3)}
\end{aligned}$$

Define:

$$\bar{v}=\left(\frac{n}{\exp(\lambda)}+\frac{1}{v}\right)^{-1} \qquad \bar{m}=\bar{v}\left(\frac{1}{\exp(\lambda)}\sum_{i=1}^{n}y_i+\frac{m}{v}\right) \qquad \text{(a.2.7.4)}$$

Then (a.2.7.3) rewrites:

$$\begin{aligned}
=\ & \frac{\mu^2}{\bar{v}}+\frac{\bar{m}^2}{\bar{v}}-2\mu\frac{\bar{m}}{\bar{v}}+\frac{1}{\exp(\lambda)}\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}-\frac{\bar{m}^2}{\bar{v}} \\
=\ & \frac{(\mu-\bar{m})^2}{\bar{v}}+\frac{1}{\exp(\lambda)}\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}-\frac{\bar{m}^2}{\bar{v}} \qquad \text{(a.2.7.5)}
\end{aligned}$$

Substitute back in (a.2.7.1):

$$\begin{aligned}
&\pi(\mu|y,\lambda) \\
\propto\;& \exp\left(-\frac{1}{2}\left[\frac{(\mu-\bar{m})^2}{\bar{v}} + \frac{1}{\exp(\lambda)}\sum_{i=1}^{n} y_i^2 + \frac{m^2}{v} - \frac{\bar{m}^2}{\bar{v}}\right]\right) \\
=\;& \exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)\exp\left(-\frac{1}{2}\left[\frac{1}{\exp(\lambda)}\sum_{i=1}^{n} y_i^2 + \frac{m^2}{v} - \frac{\bar{m}^2}{\bar{v}}\right]\right) \\
\propto\;& \exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right) \qquad \text{(a.2.7.6)}
\end{aligned}$$

derivations for equation (2.7.14)

$$\begin{aligned}
&\alpha(\lambda^{(j-1)},\lambda^{(j)}) \\
=\;& \frac{\exp(\lambda^{(j)})^{-n/2}}{\exp(\lambda^{(j-1)})^{-n/2}} \; \frac{\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda^{(j)})}\right)}{\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda^{(j-1)})}\right)} \; \frac{\exp\left(-\frac{1}{2}\frac{(\lambda^{(j)}-g)^2}{z}\right)}{\exp\left(-\frac{1}{2}\frac{(\lambda^{(j-1)}-g)^2}{z}\right)} \qquad \text{(a.2.7.7)}
\end{aligned}$$

Consider the first term:

$$\begin{aligned}
&\frac{\exp(\lambda^{(j)})^{-n/2}}{\exp(\lambda^{(j-1)})^{-n/2}} \\
=\;& \exp(\lambda^{(j)}-\lambda^{(j-1)})^{-n/2} \\
=\;& \exp\left(\frac{n}{2}(\lambda^{(j-1)}-\lambda^{(j)})\right) \qquad \text{(a.2.7.8)}
\end{aligned}$$

Consider the second term:

$$\begin{aligned}
&\frac{\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda^{(j)})}\right)}{\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda^{(j-1)})}\right)} \\
=\;& \frac{\exp\left(-\frac{1}{2}\exp(-\lambda^{(j)})\sum_{i=1}^{n}(y_i-\mu)^2\right)}{\exp\left(-\frac{1}{2}\exp(-\lambda^{(j-1)})\sum_{i=1}^{n}(y_i-\mu)^2\right)} \\
=\;& \exp\left(-\frac{1}{2}\left[\exp(-\lambda^{(j)})-\exp(-\lambda^{(j-1)})\right]\sum_{i=1}^{n}(y_i-\mu)^2\right) \\
=\;& \exp\left(\frac{1}{2}\left[\exp(-\lambda^{(j-1)})-\exp(-\lambda^{(j)})\right]\sum_{i=1}^{n}(y_i-\mu)^2\right) \qquad \text{(a.2.7.9)}
\end{aligned}$$

Consider the third term:

$$\begin{aligned}
&\frac{\exp\left(-\frac{1}{2}\frac{(\lambda^{(j)}-g)^2}{z}\right)}{\exp\left(-\frac{1}{2}\frac{(\lambda^{(j-1)}-g)^2}{z}\right)} \\
=\;& \exp\left(-\frac{1}{2}\left[\frac{(\lambda^{(j)}-g)^2-(\lambda^{(j-1)}-g)^2}{z}\right]\right) \\
=\;& \exp\left(\frac{1}{2}\left[\frac{(\lambda^{(j-1)}-g)^2-(\lambda^{(j)}-g)^2}{z}\right]\right) \qquad \text{(a.2.7.10)}
\end{aligned}$$

Substitute back in (a.2.7.7):

$$\begin{aligned}
&\alpha(\lambda^{(j-1)},\lambda^{(j)}) \\
= &\exp\left(\frac{1}{2}\left[\begin{array}{l} n(\lambda^{(j-1)}-\lambda^{(j)})+\left[\exp(-\lambda^{(j-1)})-\exp(-\lambda^{(j)})\right]\sum_{i=1}^{n}(y_i-\mu)^2 \\ +\dfrac{(\lambda^{(j-1)}-g)^2-(\lambda^{(j)}-g)^2}{z} \end{array}\right]\right) \qquad \text{(a.2.7.11)}
\end{aligned}$$

derivations for equation (2.7.21)

Rearrange the expression:

$$\begin{aligned}
&\frac{1}{f(y)} \\
\approx\ &\frac{1}{J}\sum_{j=1}^{J}\frac{g(\theta^{(j)})}{f(y|\mu^{(j)},\lambda^{(j)})\ \pi(\mu^{(j)})\ \pi(\lambda^{(j)})} \\
=\ &\frac{1}{J}\sum_{j=1}^{J}\frac{\mathbb{1}(\theta\in\hat{\Theta})\times\omega^{-1}(2\pi)^{-k/2}|\hat{\Sigma}|^{-1/2}\exp\left(-\frac{1}{2}(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\right)\mathbb{1}(\theta\in\hat{\Theta})}{(2\pi\ \exp(\lambda))^{-n/2}\ \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda)}\right)\ (2\pi v)^{-1/2}\ \exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)\ (2\pi z)^{-1/2}\ \exp\left(-\frac{1}{2}\frac{(\lambda-g)^2}{z}\right)} \\
=\ &\mathbb{1}(\theta\in\hat{\Theta})\times\omega^{-1}(2\pi)^{(n+2-k)/2}|\hat{\Sigma}|^{-1/2}(vz)^{1/2} \\
&\times\frac{1}{J}\sum_{j=1}^{J}\exp\left(\frac{1}{2}\left[n\lambda+\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda)}+\frac{(\mu-m)^2}{v}+\frac{(\lambda-g)^2}{z}-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\right]\right) \\
=\ &\mathbb{1}(\theta\in\hat{\Theta})\times(\omega J)^{-1}(2\pi)^{n/2}|\hat{\Sigma}|^{-1/2}(vz)^{1/2} \\
&\times\sum_{j=1}^{J}\exp\left(\frac{1}{2}\left[n\lambda+\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda)}+\frac{(\mu-m)^2}{v}+\frac{(\lambda-g)^2}{z}-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\right]\right) \qquad \text{(a.2.7.12)}
\end{aligned}$$

CHAPTER 8

Mathematical theory

derivations for equation (2.8.11)

The definition of an invariant distribution implies that:

$$\begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4 & \cdots\end{pmatrix}\begin{pmatrix} p+q & r & 0 & 0 & 0 & \cdots \\ p & q & r & 0 & 0 & \cdots \\ 0 & p & q & r & 0 & \cdots \\ 0 & 0 & p & q & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4 & \cdots\end{pmatrix} \tag{a.2.8.1}$$

The product with the first column of P yields:

$$\begin{aligned} & \pi_1(p+q)+\pi_2 p = \pi_1 \\ \Leftrightarrow\ & \pi_2 p = \pi_1(1-p-q) \\ \Leftrightarrow\ & \pi_2 p = \pi_1 r \\ \Leftrightarrow\ & \pi_2 = (r/p)\pi_1 \end{aligned} \tag{a.2.8.2}$$

The second column yields:

$$\begin{aligned} & \pi_1 r+\pi_2 q+\pi_3 p = \pi_2 \\ \Leftrightarrow\ & \pi_1 r+\pi_3 p = \pi_2(1-q) \\ \Leftrightarrow\ & \pi_1 r+\pi_3 p = \pi_1(r/p)(1-q) \\ \Leftrightarrow\ & \pi_1+\pi_3(p/r) = \pi_1(1-q)/p \\ \Leftrightarrow\ & \pi_3(p/r) = \pi_1(1-q-p)/p \\ \Leftrightarrow\ & \pi_3(p/r) = \pi_1(r/p) \\ \Leftrightarrow\ & \pi_3(p/r) = \pi_2 \\ \Leftrightarrow\ & \pi_3 = (r/p)\pi_2 \\ \Leftrightarrow\ & \pi_3 = (r/p)^2\pi_1 \end{aligned} \tag{a.2.8.3}$$

The third column yields:

$$
\begin{aligned}
& \pi_2 r + \pi_3 q + \pi_4 p = \pi_3 \\
\Leftrightarrow\ & \pi_2 r + \pi_4 p = \pi_3 (1-q) \\
\Leftrightarrow\ & \pi_2 r + \pi_4 p = \pi_2 (r/p)(1-q) \\
\Leftrightarrow\ & \pi_2 + \pi_4 (p/r) = \pi_2 (1-q)/p \\
\Leftrightarrow\ & \pi_4 (p/r) = \pi_2 (1-q-p)/p \\
\Leftrightarrow\ & \pi_4 (p/r) = \pi_2 (r/p) \\
\Leftrightarrow\ & \pi_4 (p/r) = \pi_3 \\
\Leftrightarrow\ & \pi_4 = (r/p)\pi_3 \\
\Leftrightarrow\ & \pi_4 = (r/p)^2 \pi_2 \\
\Leftrightarrow\ & \pi_4 = (r/p)^3 \pi_1 \qquad \text{(a.2.8.4)}
\end{aligned}
$$

Continuing this way, one obtains in general that $\pi_j = (r/p)^{j-1}\pi_1$. If an invariant distribution exists, we must have $\pi_1 + \pi_2 + \pi_3 \cdots = 1$. Hence:

$$
\begin{aligned}
& \pi_1 + \pi_2 + \pi_3 + \cdots = 1 \\
\Leftrightarrow\ & \pi_1 + (r/p)\pi_1 + (r/p)^2 \pi_1 + \cdots = 1 \\
\Leftrightarrow\ & \pi_1 (1 + (r/p) + (r/p)^2 + \cdots) = 1 \\
\Leftrightarrow\ & \pi_1 \frac{1}{1 - r/p} = 1 \\
\Leftrightarrow\ & \pi_1 = 1 - r/p \qquad \text{(a.2.8.5)}
\end{aligned}
$$

derivations for equation (2.8.16)

Start from the definition and rearrange:

$$
\begin{aligned}
\pi(y_t) &= \int \pi(y_{t-1})\ p(y_{t-1}, y_t)\ dy_{t-1} \\
&\propto \int \exp\left(-\frac{1}{2}\frac{(y_{t-1}-\mu)^2}{\sigma}\right)\ \exp\left(-\frac{1}{2}\frac{(y_t - c - \gamma y_{t-1})^2}{s}\right)\ dy_{t-1} \\
&= \int \exp\left(-\frac{1}{2}\left[\frac{(y_{t-1}-\mu)^2}{\sigma} + \frac{(y_t - c - \gamma y_{t-1})^2}{s}\right]\right)\ dy_{t-1} \\
&= \int \exp\left(-\frac{1}{2}\left[\frac{(y_{t-1}-\mu)^2(1-\gamma^2)}{s} + \frac{(y_t - c - \gamma y_{t-1})^2}{s}\right]\right)\ dy_{t-1} \\
&= \int \exp\left(-\frac{1}{2s}\left[(y_{t-1}-\mu)^2(1-\gamma^2) + (y_t - \mu(1-\gamma) - \gamma y_{t-1})^2\right]\right)\ dy_{t-1} \qquad \text{(a.2.8.6)}
\end{aligned}
$$

Consider the term within the square brackets:

$$
\begin{aligned}
& (y_{t-1}-\mu)^2(1-\gamma^2) + (y_t - \mu(1-\gamma) - \gamma y_{t-1})^2 \\
=\ & (1-\gamma^2)y_{t-1}^2 + (1-\gamma^2)\mu^2 - 2(1-\gamma^2)\mu y_{t-1} + y_t^2 + \mu^2(1-\gamma)^2 + \gamma^2 y_{t-1}^2 \\
-\ & 2\mu(1-\gamma)y_t - 2\gamma y_t y_{t-1} + 2\mu\gamma(1-\gamma)y_{t-1} \\
=\ & y_{t-1}^2 + (1-\gamma^2)\mu^2 - 2(1-\gamma^2)\mu y_{t-1} + y_t^2 + \mu^2(1-\gamma)^2 \\
-\ & 2\mu(1-\gamma)y_t - 2\gamma y_t y_{t-1} + 2\mu\gamma(1-\gamma)y_{t-1} \\
=\ & y_{t-1}^2 + (1-\gamma^2)\mu^2 - 2\mu y_{t-1} + 2\gamma^2\mu y_{t-1} + y_t^2 + \mu^2(1-\gamma)^2 \\
-\ & 2\mu(1-\gamma)y_t - 2\gamma y_t y_{t-1} + 2\mu\gamma y_{t-1} - 2\mu\gamma^2 y_{t-1} \qquad \text{(a.2.8.7)}
\end{aligned}
$$

$$
\begin{aligned}
&= y_{t-1}^2 + (1-\gamma^2)\mu^2 - 2\mu y_{t-1} + y_t^2 + \mu^2(1-\gamma)^2 - 2\mu(1-\gamma)y_t - 2\gamma y_t y_{t-1} + 2\mu\gamma y_{t-1} \\
&= y_{t-1}^2 + (1-\gamma^2)\mu^2 - 2\mu y_{t-1}(1-\gamma) + y_t^2 + \mu^2(1-\gamma)^2 - 2\mu(1-\gamma)y_t - 2\gamma y_t y_{t-1} \\
&= y_{t-1}^2 + (1-\gamma^2)\mu^2 - 2\mu y_{t-1}(1-\gamma) + y_t^2 + \mu^2(1-\gamma)^2 - 2\mu(1-\gamma^2)y_t + 2\mu(1-\gamma)\gamma y_t - 2\gamma y_t y_{t-1} \\
&= y_{t-1}^2 + (1-\gamma^2)\mu^2 - 2\mu y_{t-1}(1-\gamma) + (1-\gamma^2)y_t^2 + \gamma^2 y_t^2 + \mu^2(1-\gamma)^2 \\
&- 2\mu(1-\gamma^2)y_t + 2\mu(1-\gamma)\gamma y_t - 2\gamma y_t y_{t-1} \\
&= (1-\gamma^2)y_t^2 + (1-\gamma^2)\mu^2 - 2\mu(1-\gamma^2)y_t \\
&+ y_{t-1}^2 + \mu^2(1-\gamma)^2 + \gamma^2 y_t^2 - 2\gamma y_t y_{t-1} - 2\mu y_{t-1}(1-\gamma) + 2\mu(1-\gamma)\gamma y_t \\
&= (1-\gamma^2)y_t^2 + (1-\gamma^2)\mu^2 - 2\mu(1-\gamma^2)y_t \\
&+ y_{t-1}^2 + c^2 + \gamma^2 y_t^2 - 2\gamma y_t y_{t-1} - 2c y_{t-1} + 2c\gamma y_t \\
&= (1-\gamma^2)(y_t-\mu)^2 + (y_{t-1} - c - \gamma y_t)^2 \qquad \text{(a.2.8.8)}
\end{aligned}
$$

Substitute back in (a.2.8.6):

$$\pi(y_t) = \int \exp\left(-\frac{1}{2s}\left[(1-\gamma^2)(y_t-\mu)^2 + (y_{t-1} - c - \gamma y_t)^2\right]\right)\, dy_{t-1} \qquad \text{(a.2.8.9)}$$

and this eventually reformulates as:

$$\pi(y_t) = \exp\left(-\frac{1}{2}\frac{(y_t-\mu)^2}{\sigma}\right)\int \exp\left(-\frac{1}{2}\frac{(y_{t-1} - c - \gamma y_t)^2}{s}\right)\, dy_{t-1} \qquad \text{(a.2.8.10)}$$

PART III

Econometrics

CHAPTER 9

The linear regression model

derivations for equation (3.9.7)

Consider first β. To do so, rewrite the likelihood function as:

$$log(f(y|\beta,\sigma)) = -\frac{n}{2}\,log(2\pi) - \frac{n}{2}\,log(\sigma) - \frac{1}{2\sigma}(y'y + \beta'X'X\beta - 2\beta'X'y) \tag{a.3.9.1}$$

Then solve for the partial derivative:

$$\begin{aligned} & \frac{\partial log(f(y|\beta,\sigma))}{\partial \beta} = 0 \\ \Leftrightarrow\ & -\frac{1}{2\sigma}(2X'X\beta - 2X'y) = 0 \\ \Leftrightarrow\ & 2X'X\beta - 2X'y = 0 \\ \Leftrightarrow\ & X'X\beta - X'y = 0 \\ \Leftrightarrow\ & \beta = (X'X)^{-1}X'y \end{aligned} \tag{a.3.9.2}$$

Hence the estimate is $\hat{\beta} = (X'X)^{-1}X'y$. Consider now σ. Solve for the partial derivative:

$$\begin{aligned} & \frac{\partial log(f(y|\beta,\sigma))}{\partial \sigma} = 0 \\ \Leftrightarrow\ & -\frac{n}{2}\frac{1}{\sigma} + \frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma^2} = 0 \\ \Leftrightarrow\ & -n + \frac{(y-X\beta)'(y-X\beta)}{\sigma} = 0 \\ \Leftrightarrow\ & \frac{(y-X\beta)'(y-X\beta)}{\sigma} = n \\ \Leftrightarrow\ & \sigma = \frac{(y-X\beta)'(y-X\beta)}{n} \end{aligned} \tag{a.3.9.3}$$

This expression gives the optimum for any value of β. To obtain a global maximum we must choose the value of β that maximizes the likelihood, namely $\hat{\beta}$. Therefore, the estimate for σ is given by $\hat{\sigma} = (y - X\hat{\beta})'(y - X\hat{\beta})/n$.

derivations for equation (3.9.12)

Develop and group:

$$\begin{aligned} & \exp\left(-\frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma}\right) \times \exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right) \\ =\ & \exp\left(-\frac{1}{2}\left[(y-X\beta)'\sigma^{-1}(y-X\beta) + (\beta-b)'V^{-1}(\beta-b)\right]\right) \end{aligned} \tag{a.3.9.4}$$

Consider the term in square brackets:

$$
\begin{aligned}
& (y-X\beta)'\sigma^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b) \\
= \;& y'\sigma^{-1}y+\beta'X'\sigma^{-1}X\beta-2\beta'X'\sigma^{-1}y+\beta'V^{-1}\beta+b'V^{-1}b-2\beta'V^{-1}b \\
= \;& \beta'(V^{-1}+\sigma^{-1}X'X)\beta-2\beta'(V^{-1}b+\sigma^{-1}X'y)+b'V^{-1}b+y'\sigma^{-1}y \qquad \text{(a.3.9.5)}
\end{aligned}
$$

Substitute back in (a.3.9.4):

$$
= \;\exp\left(-\frac{1}{2}\left[\beta'(V^{-1}+\sigma^{-1}X'X)\beta-2\beta'(V^{-1}b+\sigma^{-1}X'y)+b'V^{-1}b+y'\sigma^{-1}y\right]\right) \qquad \text{(a.3.9.6)}
$$

derivations for equation (3.9.23)

Group and rearrange:

$$
\begin{aligned}
& \pi(\beta,\sigma|y) \\
\propto \;& \sigma^{-n/2}\exp\left(-\frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma}\right)\times|\sigma V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'(\sigma V)^{-1}(\beta-b)\right) \\
\times \;& \sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right) \\
= \;& \sigma^{-k/2}\exp\left(-\frac{1}{2\sigma}\left[(y-X\beta)'(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)\right]\right)\sigma^{-(\alpha+n)/2-1}\exp\left(-\frac{\delta}{2\sigma}\right) \qquad \text{(a.3.9.7)}
\end{aligned}
$$

Define:

$$
\bar{\alpha}=\alpha+n \qquad \text{(a.3.9.8)}
$$

Then:

$$
= \;\sigma^{-k/2}\exp\left(-\frac{1}{2\sigma}\left[(y-X\beta)'(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)\right]\right)\sigma^{-\bar{\alpha}/2-1}\exp\left(-\frac{\delta}{2\sigma}\right) \qquad \text{(a.3.9.9)}
$$

Consider the term between the square brackets:

$$
\begin{aligned}
& (y-X\beta)'(y-X\beta)+(\beta-b)'V^{-1}(\beta-b) \\
= \;& y'y+\beta'X'X\beta-2\beta'X'y+\beta'V^{-1}\beta+b'V^{-1}b-2\beta'V^{-1}b \\
= \;& \beta'(V^{-1}+X'X)\beta-2\beta'(V^{-1}b+X'y)+y'y+b'V^{-1}b \\
= \;& \beta'(V^{-1}+X'X)\beta-2\beta'\bar{V}^{-1}\bar{V}(V^{-1}b+X'y)+y'y+b'V^{-1}b+\bar{b}'\bar{V}^{-1}\bar{b}-\bar{b}'\bar{V}^{-1}\bar{b} \qquad \text{(a.3.9.10)}
\end{aligned}
$$

Define:

$$
\bar{V}=(V^{-1}+X'X)^{-1} \qquad \bar{b}=\bar{V}(V^{-1}b+X'y) \qquad \text{(a.3.9.11)}
$$

Then (a.3.9.10) becomes:

$$
\begin{aligned}
= \;& \beta'\bar{V}^{-1}\beta-2\beta'\bar{V}^{-1}\bar{b}+y'y+b'V^{-1}b+\bar{b}'\bar{V}^{-1}\bar{b}-\bar{b}'\bar{V}^{-1}\bar{b} \\
= \;& (\beta'\bar{V}^{-1}\beta-2\beta'\bar{V}\bar{b}+\bar{b}'\bar{V}^{-1}\bar{b})+y'y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b} \\
= \;& (\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+y'y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b} \qquad \text{(a.3.9.12)}
\end{aligned}
$$

Substitute back in (a.3.9.9):

$$
= \;\sigma^{-k/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'(\sigma\bar{V})^{-1}(\beta-\bar{b})\right)\sigma^{-\bar{\alpha}/2-1}\exp\left(-\frac{\delta+y'y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}}{2\sigma}\right) \qquad \text{(a.3.9.13)}
$$

Define:

$$\bar{\delta} = \delta + y'y + b'V^{-1}b - \bar{b}'\bar{V}^{-1}\bar{b} \tag{a.3.9.14}$$

Then (a.3.9.13) eventually rewrites:

$$\pi(\beta,\sigma|y) \propto \sigma^{-k/2} \exp\left(-\frac{1}{2}(\beta-\bar{b})'(\sigma\bar{V})^{-1}(\beta-\bar{b})\right) \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right) \tag{a.3.9.15}$$

derivations for equation (3.9.28)

Rearrange:

$$\begin{aligned}
&\Gamma\left(\frac{\bar{\alpha}+k}{2}\right)\left(\frac{\bar{\delta}+(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})}{2}\right)^{-\frac{\bar{\alpha}+k}{2}} \\
\propto\ &\left(\frac{\bar{\delta}+(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})}{2}\right)^{-\frac{\bar{\alpha}+k}{2}} \\
\propto\ &\left(\bar{\delta}+(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)^{-\frac{\bar{\alpha}+k}{2}} \\
\propto\ &\left(1+(\beta-\bar{b})'(\bar{\delta}\bar{V})^{-1}(\beta-\bar{b})\right)^{-\frac{\bar{\alpha}+k}{2}} \\
\propto\ &\left(1+\frac{1}{\bar{\alpha}}(\beta-\bar{b})'(\bar{\delta}\bar{V}/\bar{\alpha})^{-1}(\beta-\bar{b})\right)^{-\frac{\bar{\alpha}+k}{2}}
\end{aligned} \tag{a.3.9.16}$$

derivations for equation (3.9.38)

Rearrange the likelihood function:

$$\begin{aligned}
&f(y|\beta,\sigma) \\
=\ &(2\pi)^{-n/2}\ |\sigma W|^{-1/2} \exp\left(-\frac{1}{2}(y-X\beta)'(\sigma W)^{-1}(y-X\beta)\right) \\
=\ &(2\pi\sigma)^{-n/2}\ |W|^{-1/2} \exp\left(-\frac{1}{2}\frac{(y-X\beta)'W^{-1}(y-X\beta)}{\sigma}\right)
\end{aligned} \tag{a.3.9.17}$$

This reformulates further as:

$$\begin{aligned}
=\ &(2\pi\sigma)^{-n/2}\left(\prod_{i=1}^{n} w_i^{-1/2}\right) \exp\left(-\frac{1}{2}\frac{(y-X\beta)'\ diag(\exp(-Z\gamma))\ (y-X\beta)}{\sigma}\right) \\
=\ &(2\pi\sigma)^{-n/2}\left(\prod_{i=1}^{n}\exp(z_i'\gamma)\right)^{-1/2} \exp\left(-\frac{1}{2}\frac{(y-X\beta)'\ diag(\exp(-Z\gamma))\ (y-X\beta)}{\sigma}\right) \\
=\ &(2\pi\sigma)^{-n/2}\left(\exp\left(\sum_{i=1}^{n} z_i'\gamma\right)\right)^{-1/2} \exp\left(-\frac{1}{2}\frac{(y-X\beta)'\ diag(\exp(-Z\gamma))\ (y-X\beta)}{\sigma}\right) \\
=\ &(2\pi\sigma)^{-n/2}\left(\exp\left(1_n'Z\gamma\right)\right)^{-1/2} \exp\left(-\frac{1}{2}\frac{(y-X\beta)'\ diag(\exp(-Z\gamma))\ (y-X\beta)}{\sigma}\right) \\
=\ &(2\pi\sigma)^{-n/2} \exp\left(-\frac{1}{2}1_n'Z\gamma\right) \exp\left(-\frac{1}{2}\frac{(y-X\beta)'\ diag(\exp(-Z\gamma))\ (y-X\beta)}{\sigma}\right) \\
=\ &(2\pi\sigma)^{-n/2} \exp\left(-\frac{1}{2}\left[1_n'Z\gamma+(y-X\beta)'\ diag(\exp(-Z\gamma))\ (y-X\beta)/\sigma\right]\right)
\end{aligned} \tag{a.3.9.18}$$

derivations for equation (3.9.44)

Rearrange the terms:

$$\begin{aligned}
& \pi(\beta|y,\sigma,w) \\
= \; & \exp\left(-\frac{1}{2}\frac{(y-X\beta)'W^{-1}(y-X\beta)}{\sigma}\right)\times\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right) \\
= \; & \exp\left(-\frac{1}{2}\left[(y-X\beta)'(\sigma W)^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)\right)\right]
\end{aligned} \tag{a.3.9.19}$$

Consider the term in square brackets and complete the squares:

$$\begin{aligned}
& (y-X\beta)'(\sigma W)^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b) \\
= \; & y'(\sigma W)^{-1}y+\beta'X'(\sigma W)^{-1}X\beta-2\beta'X'(\sigma W)^{-1}y+\beta'V^{-1}\beta+b'V^{-1}b-2\beta'V^{-1}b \\
= \; & \beta'(V^{-1}+\sigma^{-1}X'W^{-1}X)\beta-2\beta'(V^{-1}b+\sigma^{-1}X'W^{-1}y)+y'(\sigma W)^{-1}y+b'V^{-1}b \\
= \; & \beta'(V^{-1}+\sigma^{-1}X'W^{-1}X)\beta-2\beta'\bar{V}^{-1}\bar{V}(V^{-1}b+\sigma^{-1}X'W^{-1}y) \\
& +y'(\sigma W)^{-1}y+b'V^{-1}b+\bar{b}'\bar{V}^{-1}\bar{b}-\bar{b}'\bar{V}^{-1}\bar{b}
\end{aligned} \tag{a.3.9.20}$$

Define:

$$\bar{V}=(V^{-1}+\sigma^{-1}X'W^{-1}X)^{-1} \qquad \bar{b}=\bar{V}(V^{-1}b+\sigma^{-1}X'W^{-1}y) \tag{a.3.9.21}$$

Then (a.3.9.20) rewrites:

$$\begin{aligned}
= \; & \beta'\bar{V}^{-1}\beta-2\beta'\bar{V}^{-1}\bar{b}+\bar{b}'\bar{V}^{-1}\bar{b}+y'(\sigma W)^{-1}y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b} \\
= \; & (\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+y'(\sigma W)^{-1}y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}
\end{aligned} \tag{a.3.9.22}$$

Substitute back in (a.3.9.19):

$$\begin{aligned}
& \pi(\beta|y,\sigma,w) \\
= \; & \exp\left(-\frac{1}{2}\left[(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+y'(\sigma W)^{-1}y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}\right]\right) \\
= \; & \exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)\exp\left(-\frac{1}{2}\left[y'(\sigma W)^{-1}y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}\right]\right) \\
\propto \; & \exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)
\end{aligned} \tag{a.3.9.23}$$

derivations for equation (3.9.69)

Rearrange the terms:

$$\begin{aligned}
& \pi(\phi|y,\beta,\sigma) \\
\propto \; & \exp\left(-\frac{1}{2}\frac{(\varepsilon-E\phi)'(\varepsilon-E\phi)}{\sigma}\right)\times\exp\left(-\frac{1}{2}(\phi-p)'H^{-1}(\phi-p)\right) \\
= \; & \exp\left(-\frac{1}{2}\left[(\varepsilon-E\phi)'\sigma^{-1}(\varepsilon-E\phi)+(\phi-p)'H^{-1}(\phi-p)\right]\right)
\end{aligned} \tag{a.3.9.24}$$

Consider the term in square brackets and complete the squares:

$$\begin{aligned}
& (\varepsilon - E\phi)'\sigma^{-1}(\varepsilon - E\phi) + (\phi - p)'H^{-1}(\phi - p) \\
= \; & \varepsilon'\sigma^{-1}\varepsilon + \phi'E'\sigma^{-1}E\phi - 2\phi'E'\sigma^{-1}\varepsilon + \phi'H^{-1}\phi + p'H^{-1}p - 2\phi'H^{-1}p \\
= \; & \phi'(H^{-1} + \sigma^{-1}E'E)\phi - 2\phi'(H^{-1}p + \sigma^{-1}E'\varepsilon) + \varepsilon'\sigma^{-1}\varepsilon + p'H^{-1}p \\
= \; & \phi'(H^{-1} + \sigma^{-1}E'E)\phi - 2\phi'\bar{H}^{-1}\bar{H}(H^{-1}p + \sigma^{-1}E'\varepsilon) + \varepsilon'\sigma^{-1}\varepsilon + p'H^{-1}p + \bar{p}'\bar{H}^{-1}\bar{p} - \bar{p}'\bar{H}^{-1}\bar{p}
\end{aligned} \tag{a.3.9.25}$$

Define:

$$\bar{H} = (H^{-1} + \sigma^{-1}E'E)^{-1} \qquad \bar{p} = \bar{H}(H^{-1}p + \sigma^{-1}E'\varepsilon) \tag{a.3.9.26}$$

Then (a.3.9.25) becomes:

$$\begin{aligned}
= \; & \phi'\bar{H}^{-1}\phi - 2\phi'\bar{H}^{-1}\bar{p} + \bar{p}'\bar{H}^{-1}\bar{p} + \varepsilon'\sigma^{-1}\varepsilon + p'H^{-1}p - \bar{p}'\bar{H}^{-1}\bar{p} \\
= \; & (\phi - \bar{p})'\bar{H}^{-1}(\phi - \bar{p}) + \varepsilon'\sigma^{-1}\varepsilon + p'H^{-1}p - \bar{p}'\bar{H}^{-1}\bar{p}
\end{aligned} \tag{a.3.9.27}$$

Substitute back in (a.3.9.24) to obtain:

$$\begin{aligned}
& \pi(\phi|y,\beta,\sigma) \\
= \; & \exp\left(-\frac{1}{2}\left[(\phi - \bar{p})'\bar{H}^{-1}(\phi - \bar{p}) + \varepsilon'\sigma^{-1}\varepsilon + p'H^{-1}p - \bar{p}'\bar{H}^{-1}\bar{p}\right]\right) \\
= \; & \exp\left(-\frac{1}{2}(\phi - \bar{p})'\bar{H}^{-1}(\phi - \bar{p})\right)\exp\left(-\frac{1}{2}\left[\varepsilon'\sigma^{-1}\varepsilon + p'H^{-1}p - \bar{p}'\bar{H}^{-1}\bar{p}\right]\right) \\
\propto \; & \exp\left(-\frac{1}{2}(\phi - \bar{p})'\bar{H}^{-1}(\phi - \bar{p})\right)
\end{aligned} \tag{a.3.9.28}$$

CHAPTER 10

Applications with the linear regression model

derivations for equation (3.10.4)

Rearrange the expression:

$$\begin{aligned}
& f(\hat{y}|y) \\
\propto & \int \exp\left(-\frac{1}{2}\frac{(\hat{y}-\hat{X}\beta)'(\hat{y}-\hat{X}\beta)}{\sigma}\right) \exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right) d\beta \\
= & \int \exp\left(-\frac{1}{2}\left[\sigma^{-1}(\hat{y}-\hat{X}\beta)'(\hat{y}-\hat{X}\beta)+(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right]\right) d\beta \qquad \text{(a.3.10.1)}
\end{aligned}$$

Consider the term in square brackets:

$$\begin{aligned}
& \sigma^{-1}(\hat{y}-\hat{X}\beta)'(\hat{y}-\hat{X}\beta)+(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b}) \\
= & \sigma^{-1}\hat{y}'\hat{y}+\sigma^{-1}\beta'\hat{X}'\hat{X}\beta-2\sigma^{-1}\beta'\hat{X}'\hat{y}+\beta'\bar{V}^{-1}\beta+\bar{b}'\bar{V}^{-1}\bar{b}-2\beta'\bar{V}^{-1}\bar{b} \\
= & \beta'(\bar{V}^{-1}+\sigma^{-1}\hat{X}'\hat{X})\beta-2\beta'(\bar{V}^{-1}\bar{b}+\sigma^{-1}\hat{X}'\hat{y})+\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b} \\
= & \beta'(\bar{V}^{-1}+\sigma^{-1}\hat{X}'\hat{X})\beta-2\beta'\hat{V}^{-1}\hat{V}(\bar{V}^{-1}\bar{b}+\sigma^{-1}\hat{X}'\hat{y})+\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}+\hat{b}'\hat{V}^{-1}\hat{b}-\hat{b}'\hat{V}^{-1}\hat{b} \qquad \text{(a.3.10.2)}
\end{aligned}$$

Define:

$$\hat{V}=(\bar{V}^{-1}+\sigma^{-1}\hat{X}'\hat{X})^{-1} \qquad \hat{b}=\hat{V}(\bar{V}^{-1}\bar{b}+\sigma^{-1}\hat{X}'\hat{y}) \qquad \text{(a.3.10.3)}$$

Then (a.3.10.2) becomes:

$$\begin{aligned}
= & \beta'\hat{V}^{-1}\beta-2\beta'\hat{V}^{-1}\hat{b}+\hat{b}'\hat{V}^{-1}\hat{b}+\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\hat{b}'\hat{V}^{-1}\hat{b} \\
= & (\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})+\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\hat{b}'\hat{V}^{-1}\hat{b} \qquad \text{(a.3.10.4)}
\end{aligned}$$

Substituting back in (a.3.10.1):

$$\begin{aligned}
& f(\hat{y}|y) \\
\propto & \int \exp\left(-\frac{1}{2}\left[(\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})+\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\hat{b}'\hat{V}^{-1}\hat{b}\right]\right) d\beta \\
= & \int \exp\left(-\frac{1}{2}(\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})\right)\exp\left(-\frac{1}{2}\left[\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\hat{b}'\hat{V}^{-1}\hat{b}\right]\right) d\beta \\
= & \exp\left(-\frac{1}{2}\left[\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\hat{b}'\hat{V}^{-1}\hat{b}\right]\right)\int \exp\left(-\frac{1}{2}(\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})\right) d\beta \\
\propto & \exp\left(-\frac{1}{2}\left[\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\hat{b}'\hat{V}^{-1}\hat{b}\right]\right) \qquad \text{(a.3.10.5)}
\end{aligned}$$

Consider the term in square brackets:

$$\begin{aligned}
& \sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\hat{b}'\hat{V}^{-1}\hat{b} \\
=\ & \sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-(\bar{V}^{-1}\bar{b}+\sigma^{-1}\hat{X}'\hat{y})'\hat{V}\hat{V}^{-1}\hat{V}(\bar{V}^{-1}\bar{b}+\sigma^{-1}\hat{X}'\hat{y}) \\
=\ & \sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-(\bar{V}^{-1}\bar{b}+\sigma^{-1}\hat{X}'\hat{y})'\hat{V}(\bar{V}^{-1}\bar{b}+\sigma^{-1}\hat{X}'\hat{y}) \\
=\ & \sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\bar{b}'\bar{V}^{-1}\hat{V}\bar{V}^{-1}\bar{b}-\sigma^{-2}\hat{y}'\hat{X}\hat{V}\hat{X}'\hat{y}-2\sigma^{-1}\hat{y}'\hat{X}\hat{V}\bar{V}^{-1}\bar{b} \\
=\ & \hat{y}'(\sigma^{-1}I_m-\sigma^{-2}\hat{X}\hat{V}\hat{X}')\hat{y}-\bar{b}'(\bar{V}^{-1}-\bar{V}^{-1}\hat{V}\bar{V}^{-1})\bar{b}-2\sigma^{-1}\hat{y}'\hat{X}\hat{V}\bar{V}^{-1}\bar{b} \qquad \text{(a.3.10.6)}
\end{aligned}$$

In what follows, we make use of property m.13 (the Sherman-Woodbury-Morrison identity): $(A+BDC)^{-1}=A^{-1}-A^{-1}B(D^{-1}+CA^{-1}B)^{-1}CA^{-1}$.

Consider the central part of the second term in (a.3.10.6). Rearrange and use the identity twice to obtain:

$$\begin{aligned}
& \bar{V}^{-1}-\bar{V}^{-1}\hat{V}\bar{V}^{-1} \\
=\ & \bar{V}^{-1}-\bar{V}^{-1}(\bar{V}^{-1}+\sigma^{-1}\hat{X}'\hat{X})^{-1}\bar{V}^{-1} \\
=\ & (\bar{V}+\sigma(\hat{X}'\hat{X})^{-1})^{-1} \\
=\ & \sigma^{-1}\hat{X}'\hat{X}-\sigma^{-1}\hat{X}'\hat{X}(\sigma^{-1}\hat{X}'\hat{X}+\bar{V}^{-1})^{-1}\sigma^{-1}\hat{X}'\hat{X} \\
=\ & \sigma^{-1}\hat{X}'\hat{X}-\sigma^{-2}\hat{X}'\hat{X}\hat{V}\hat{X}'\hat{X} \\
=\ & \hat{X}'(\sigma^{-1}I_m-\sigma^{-2}\hat{X}\hat{V}\hat{X}')\hat{X} \qquad \text{(a.3.10.7)}
\end{aligned}$$

Consider finally the contral part of the third term in (a.3.10.6). We note that $\hat{V}(\bar{V}^{-1}+\sigma^{-1}\hat{X}'\hat{X})=I_m$ so $\hat{V}\bar{V}^{-1}=I_m-\hat{V}\sigma^{-1}\hat{X}'\hat{X}$. Following:

$$\hat{X}\hat{V}\bar{V}^{-1}=\hat{X}-\hat{X}\hat{V}\sigma^{-1}\hat{X}'\hat{X}=(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{X} \qquad \text{(a.3.10.8)}$$

Substitute (a.3.10.7) and (a.3.10.8) back in (a.3.10.6) to obtain:

$$\begin{aligned}
=\ & \hat{y}'(\sigma^{-1}I_m-\sigma^{-2}\hat{X}\hat{V}\hat{X}')\hat{y}-\bar{b}'\hat{X}'(\sigma^{-1}I_m-\sigma^{-2}\hat{X}\hat{V}\hat{X}')\hat{X}\bar{b}-2\sigma^{-1}\hat{y}'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{X}\bar{b} \\
=\ & \sigma^{-1}\hat{y}'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{y}-\sigma^{-1}\bar{b}'\hat{X}'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{X}\bar{b}-2\sigma^{-1}\hat{y}'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{X}\bar{b} \\
=\ & \sigma^{-1}\left[\hat{y}'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{y}-\bar{b}'\hat{X}'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{X}\bar{b}-2\hat{y}'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{X}\bar{b}\right] \\
=\ & \sigma^{-1}(\hat{y}-\hat{X}\bar{b})'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')(\hat{y}-\hat{X}\bar{b}) \qquad \text{(a.3.10.9)}
\end{aligned}$$

We use one last time the Sherman-Woodbury-Morrison identity on the central term to obtain:

$$I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}'=I_m-\sigma^{-1}\hat{X}(\bar{V}^{-1}+\sigma^{-1}\hat{X}'\hat{X})^{-1}\hat{X}'=(I_m+\sigma^{-1}\hat{X}\bar{V}\hat{X}')^{-1} \qquad \text{(a.3.10.10)}$$

Substituting back in (a.3.10.9):

$$\begin{aligned}
=\ & \sigma^{-1}(\hat{y}-\hat{X}\bar{b})'(I_m+\sigma^{-1}\hat{X}\bar{V}\hat{X}')^{-1}(\hat{y}-\hat{X}\bar{b}) \\
=\ & (\hat{y}-\hat{X}\bar{b})'(\sigma I_m+\hat{X}\bar{V}\hat{X}')^{-1}(\hat{y}-\hat{X}\bar{b}) \qquad \text{(a.3.10.11)}
\end{aligned}$$

Eventually substituting back in (a.3.10.5), we conclude:

$$f(\hat{y}|y)\propto\exp\left(-\frac{1}{2}(\hat{y}-\hat{X}\bar{b})'(\sigma I_m+\hat{X}\bar{V}\hat{X}')^{-1}(\hat{y}-\hat{X}\bar{b})\right) \qquad \text{(a.3.10.12)}$$

derivations for equation (3.10.6)

Rearrange the expression:

$$
\begin{aligned}
& f(\hat{y}|y) \\
\propto\ & \int\int \sigma^{-m/2}\exp\left(-\frac{1}{2}\frac{(\hat{y}-\hat{X}\beta)'(\hat{y}-\hat{X}\beta)}{\sigma}\right)\ \exp\left(-\frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma}\right) \\
\times\ & \sigma^{-k/2}\exp\left(-\frac{1}{2}(\beta-b)'(\sigma V)^{-1}(\beta-b)\right)\times\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right) \\
=\ & \int\int \sigma^{-k/2}\exp\left(-\frac{1}{2\sigma}\left[(\hat{y}-\hat{X}\beta)'(\hat{y}-\hat{X}\beta)+(y-X\beta)'(y-X\beta)+(\beta-b)'(\sigma V)^{-1}(\beta-b)+\delta\right]\right) \\
\times\ & \sigma^{-(\alpha+n+m)/2-1}d\beta d\sigma
\end{aligned}
\tag{a.3.10.13}
$$

Consider the term in the square bracket:

$$
\begin{aligned}
& (\hat{y}-\hat{X}\beta)'(\hat{y}-\hat{X}\beta)+(y-X\beta)'(y-X\beta)+(\beta-b)'(\sigma V)^{-1}(\beta-b)+\delta \\
=\ & \hat{y}'\hat{y}+\beta'\hat{X}'\hat{X}\beta-2\beta'\hat{X}'\hat{y}+y'y+\beta'X'X\beta-2\beta'X'y+\beta'V^{-1}\beta+b'V^{-1}b-2\beta'V^{-1}b+\delta \\
=\ & (\delta+\hat{y}'\hat{y}+y'y+b'V^{-1}b)+\beta'(V^{-1}+X'X+\hat{X}'\hat{X})\beta-2\beta'(V^{-1}b+X'y+\hat{X}'\hat{y}) \\
=\ & (\delta+\hat{y}'\hat{y}+y'y+b'V^{-1}b)+\beta'(V^{-1}+X'X+\hat{X}'\hat{X})\beta-2\beta'\hat{V}^{-1}\hat{V}(V^{-1}b+X'y+\hat{X}'\hat{y})+\hat{b}'\hat{V}^{-1}\hat{b}-\hat{b}'\hat{V}^{-1}\hat{b} \\
=\ & (\delta+\hat{y}'\hat{y}+y'y+b'V^{-1}b-\hat{b}'\hat{V}^{-1}\hat{b})+\beta'(V^{-1}+X'X+\hat{X}'\hat{X})\beta-2\beta'\hat{V}^{-1}\hat{V}(V^{-1}b+X'y+\hat{X}'\hat{y})+\hat{b}'\hat{V}^{-1}\hat{b}
\end{aligned}
\tag{a.3.10.14}
$$

Define:

$$
\hat{\delta}=\delta+\hat{y}'\hat{y}+y'y+b'V^{-1}b-\hat{b}'\hat{V}^{-1}\hat{b} \qquad \hat{V}=(V^{-1}+X'X+\hat{X}'\hat{X})^{-1} \qquad \hat{b}=\hat{V}(V^{-1}b+X'y+\hat{X}'\hat{y})
\tag{a.3.10.15}
$$

Then (a.3.10.14) rewrites:

$$
\begin{aligned}
=\ & \hat{\delta}+\beta'\hat{V}^{-1}\beta-2\beta'\hat{V}^{-1}\hat{b}+\hat{b}'\hat{V}^{-1}\hat{b} \\
=\ & \hat{\delta}+(\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})
\end{aligned}
\tag{a.3.10.16}
$$

Substitute back in (a.3.10.13):

$$
\begin{aligned}
& f(\hat{y}|y) \\
\propto\ & \int\int \sigma^{-k/2}\exp\left(-\frac{1}{2\sigma}\left[\hat{\delta}+(\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})\right]\right)\ \sigma^{-(\alpha+n+m)/2-1}d\beta d\sigma \\
=\ & \int\int \sigma^{-k/2}\exp\left(-\frac{1}{2\sigma}(\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})\right)d\beta\ \sigma^{-(\alpha+n+m)/2-1}\exp\left(-\frac{\hat{\delta}}{2\sigma}\right)d\sigma \\
=\ & \int\int \sigma^{-k/2}\exp\left(-\frac{1}{2\sigma}(\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})\right)d\beta\ \sigma^{-(\bar{\alpha}+m)/2-1}\exp\left(-\frac{\hat{\delta}}{2\sigma}\right)d\sigma
\end{aligned}
\tag{a.3.10.17}
$$

with:

$$
\hat{\alpha}=\alpha+n+m \tag{a.3.10.18}
$$

The first term is the kernel of a multivariate normal distribution; integration hence yields a constant:

$$
= \int \sigma^{-\hat{\alpha}/2-1}\exp\left(-\frac{\hat{\delta}}{2\sigma}\right)d\sigma \tag{a.3.10.19}
$$

The remaining term is the krenel of an inverse gamma distribution; integration thus yields the reciprocal of the normalization constant:

$$\begin{aligned}
&= \Gamma(\hat{\alpha}/2)(\hat{\delta}/2)^{-\hat{\alpha}/2} \\
&\propto (\hat{\delta}/2)^{-\hat{\alpha}/2} \\
&\propto \hat{\delta}^{-\hat{\alpha}/2} \\
&= (\delta + \hat{y}'\hat{y} + y'y + b'V^{-1}b - \hat{b}'\hat{V}^{-1}\hat{b})^{-\hat{\alpha}/2} \\
&= (\delta + \hat{y}'\hat{y} + y'y + b'V^{-1}b - (V^{-1}b + X'y + \hat{X}'\hat{y})'\hat{V}'\hat{V}^{-1}\hat{V}(V^{-1}b + X'y + \hat{X}'\hat{y}))^{-\hat{\alpha}/2} \\
&= (\delta + \hat{y}'\hat{y} + y'y + b'V^{-1}b - (V^{-1}b + X'y + \hat{X}'\hat{y})'\hat{V}(V^{-1}b + X'y + \hat{X}'\hat{y}))^{-\hat{\alpha}/2} \qquad \text{(a.3.10.20)}
\end{aligned}$$

Define:

$$\tilde{b} = V^{-1}b + X'y \qquad \text{(a.3.10.21)}$$

Then (a.3.10.20) rewrites:

$$\begin{aligned}
&= (\delta + \hat{y}'\hat{y} + y'y + b'V^{-1}b - (\tilde{b} + \hat{X}'\hat{y})'\hat{V}(\tilde{b} + \hat{X}'\hat{y}))^{-\hat{\alpha}/2} \\
&= (\delta + \hat{y}'\hat{y} + y'y + b'V^{-1}b - \tilde{b}'\hat{V}\tilde{b} - \hat{y}'\hat{X}\hat{V}\hat{X}'\hat{y} - 2\hat{y}'\hat{X}\hat{V}\tilde{b})^{-\hat{\alpha}/2} \\
&= (\delta + y'y + b'V^{-1}b + \hat{y}'(I_m - \hat{X}\hat{V}\hat{X}')\hat{y} - \tilde{b}'\hat{V}\tilde{b} - 2\hat{y}'\hat{X}\hat{V}\tilde{b})^{-\hat{\alpha}/2} \\
&= ([\delta + y'y + b'V^{-1}b - \tilde{b}'\hat{V}\tilde{b} - \ddot{y}'\ddot{V}^{-1}\ddot{y}] + \hat{y}'(I_m - \hat{X}\hat{V}\hat{X}')\hat{y} - 2\hat{y}'\ddot{V}^{-1}\ddot{V}\hat{X}\hat{V}\tilde{b} + \ddot{y}'\ddot{V}^{-1}\ddot{y})^{-\hat{\alpha}/2} \qquad \text{(a.3.10.22)}
\end{aligned}$$

Define:

$$\ddot{\delta} = \delta + y'y + b'V^{-1}b - \tilde{b}'\hat{V}\tilde{b} - \ddot{y}'\ddot{V}^{-1}\ddot{y} \qquad \ddot{V} = (I_m - \hat{X}\hat{V}\hat{X}')^{-1} \qquad \ddot{y} = \ddot{V}\hat{X}\hat{V}\tilde{b} \qquad \text{(a.3.10.23)}$$

Then (a.3.10.22) rewrites:

$$\begin{aligned}
&= (\ddot{\delta} + \hat{y}'\ddot{V}^{-1}\hat{y} - 2\hat{y}'\ddot{V}^{-1}\ddot{y} + \ddot{y}'\ddot{V}^{-1}\ddot{y})^{-\hat{\alpha}/2} \\
&= (\ddot{\delta} + (\hat{y} - \ddot{y})'\ddot{V}^{-1}(\hat{y} - \ddot{y}))^{-\hat{\alpha}/2} \\
&= \ddot{\delta}^{-\hat{\alpha}/2}(1 + (\hat{y} - \ddot{y})'[\ddot{\delta}\ddot{V}]^{-1}(\hat{y} - \ddot{y}))^{-\hat{\alpha}/2} \\
&\propto (1 + (\hat{y} - \ddot{y})'[\ddot{\delta}\ddot{V}]^{-1}(\hat{y} - \ddot{y}))^{-\hat{\alpha}/2} \\
&= \left(1 + \frac{1}{\bar{\alpha}}(\hat{y} - \ddot{y})'[\ddot{\delta}\ddot{V}/\bar{\alpha}]^{-1}(\hat{y} - \ddot{y})\right)^{-(\bar{\alpha}+m)/2} \qquad \text{(a.3.10.24)}
\end{aligned}$$

Thus we finally conclude:

$$f(\hat{y}|y) \propto \left(1 + \frac{1}{\bar{\alpha}}(\hat{y} - \ddot{y})'[\ddot{\delta}\ddot{V}/\bar{\alpha}]^{-1}(\hat{y} - \ddot{y})\right)^{-(\bar{\alpha}+m)/2} \qquad \text{(a.3.10.25)}$$

Finally, reformulate the messy terms. First, reformulate $\ddot{V}$. For this, we make again use of property m.13 (the Sherman-Woodbury-Morrison identity): $(A + BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}$.

Then, starting from (a.3.10.23):

$$\begin{aligned}
&\ddot{V} \\
&= (I_m - \hat{X}\hat{V}\hat{X}')^{-1} \\
&= (I_m - \hat{X}(V^{-1} + X'X + \hat{X}'\hat{X})^{-1}\hat{X}')^{-1} \\
&= I_m + \hat{X}(V^{-1} + X'X)^{-1}\hat{X}' \\
&= I_m + \hat{X}\bar{V}\hat{X}' \qquad \text{(a.3.10.26)}
\end{aligned}$$

Now consider the term $\ddot{y}$. Start from:

$$
\begin{aligned}
& \ddot{V}\hat{X}\hat{V} \\
= \ & (I_m + \hat{X}\bar{V}\hat{X}')\hat{X}\hat{V} \\
= \ & \hat{X}\hat{V} + \hat{X}\bar{V}\hat{X}'\hat{X}\hat{V} \\
= \ & \hat{X}(\hat{V} + \bar{V}\hat{X}'\hat{X}\hat{V}) \\
= \ & \hat{X}(I_m + \bar{V}\hat{X}'\hat{X})\hat{V}
\end{aligned}
\tag{a.3.10.27}
$$

We then note that (a.3.10.15) implies:

$$
\hat{V} = (V^{-1} + X'X + \hat{X}'\hat{X})^{-1} \Leftrightarrow \hat{V} = (\bar{V}^{-1} + \hat{X}'\hat{X})^{-1} \tag{a.3.10.28}
$$

Hence:

$$
\begin{aligned}
= \ & \hat{X}(I_m + \bar{V}\hat{X}'\hat{X})(\bar{V}^{-1} + \hat{X}'\hat{X})^{-1} \\
= \ & \hat{X}\bar{V}(\bar{V}^{-1} + \hat{X}'\hat{X})(\bar{V}^{-1} + \hat{X}'\hat{X})^{-1} \\
= \ & \hat{X}\bar{V}
\end{aligned}
\tag{a.3.10.29}
$$

Using this result in (a.3.10.23), and combining with definition (a.3.10.21), we obtain:

$$
\ddot{y} = \ddot{V}\hat{X}\hat{V}\tilde{b} = \hat{X}\bar{V}\tilde{b} = \hat{X}\bar{b} \tag{a.3.10.30}
$$

Finally, reformulate $\ddot{\delta}$. First, note that:

$$
\begin{aligned}
& \tilde{b}'\hat{V}\tilde{b} + \ddot{y}'\ddot{V}^{-1}\ddot{y} \\
= \ & \bar{b}'\bar{V}^{-1}\hat{V}\bar{V}^{-1}\bar{b} + \bar{b}'\hat{X}'(I_m - \hat{X}\hat{V}\hat{X}')\hat{X}\bar{b} \\
= \ & \bar{b}'[(\hat{V}^{-1} - \hat{X}'\hat{X})'\hat{V}(\hat{V}^{-1} - \hat{X}'\hat{X}) + \hat{X}'\hat{X} - \hat{X}'\hat{X}\hat{V}\hat{X}'\hat{X}]\bar{b} \\
= \ & \bar{b}'[(\hat{V}^{-1} - \hat{X}'\hat{X})'(I_k - \hat{V}\hat{X}'\hat{X}) + \hat{X}'\hat{X} - \hat{X}'\hat{X}\hat{V}\hat{X}'\hat{X}]\bar{b} \\
= \ & \bar{b}'[\hat{V}^{-1} - \hat{X}'\hat{X} - \hat{X}'\hat{X} + \hat{X}'\hat{X}\hat{V}\hat{X}'\hat{X} + \hat{X}'\hat{X} - \hat{X}'\hat{X}\hat{V}\hat{X}'\hat{X}]\bar{b} \\
= \ & \bar{b}'[\hat{V}^{-1} - \hat{X}'\hat{X}]\bar{b} \\
= \ & \bar{b}'[\bar{V}^{-1} + \hat{X}'\hat{X} - \hat{X}'\hat{X}]\bar{b} \\
= \ & \bar{b}'\bar{V}^{-1}\bar{b}
\end{aligned}
\tag{a.3.10.31}
$$

Substituting this in (a.3.10.23) to obtain:

$$
\ddot{\delta} \ = \ \delta + y'y + b'V^{-1}b - \tilde{b}'\hat{V}\tilde{b} - \ddot{y}'\ddot{V}^{-1}\ddot{y} \ = \ \delta + y'y + b'V^{-1}b - \bar{b}'\bar{V}^{-1}\bar{b} \ = \ \bar{\delta} \tag{a.3.10.32}
$$

Eventually substituting for (a.3.10.26), (a.3.10.30) and (a.3.10.32) in (a.3.10.25) yields:

$$
f(\hat{y}|y) \propto \left(1 + \frac{1}{\bar{\alpha}}(\hat{y} - \hat{X}\bar{b})'[\bar{\delta}(I_m + \hat{X}\bar{V}\hat{X}')/\bar{\alpha}]^{-1}(\hat{y} - \hat{X}\bar{b})\right)^{-(\bar{\alpha}+m)/2} \tag{a.3.10.33}
$$

derivations for equation (3.10.10)

The log likelihood function is given by:

$$\log(f(y|\beta,\sigma)) = -\frac{n}{2}\ \log(2\pi) - \frac{n}{2}\ \log(\sigma) - \frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma} \tag{a.3.10.34}$$

The function is estimated at the maximum likelihood values. Hence $\beta = \hat{\beta}$ and $\sigma = \hat{\sigma} = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{n}$.

Substituting in (a.3.10.34):

$$\begin{aligned}
& \log(f(y|\hat{\beta},\hat{\sigma})) \\
= & -\frac{n}{2}\ \log(2\pi) - \frac{n}{2}\ \log(\hat{\sigma}) - \frac{1}{2}\frac{(y-X\hat{\beta})'(y-X\hat{\beta})}{\hat{\sigma}} \\
= & -\frac{n}{2}\ \log(2\pi) - \frac{n}{2}\ \log\left(\frac{\hat{\varepsilon}'\hat{\varepsilon}}{n}\right) - \frac{1}{2}\frac{n\ \hat{\varepsilon}'\hat{\varepsilon}}{\hat{\varepsilon}'\hat{\varepsilon}} \\
= & -\frac{n}{2}\ \log(2\pi) - \frac{n}{2}\ \log(\hat{\varepsilon}'\hat{\varepsilon}/n) - \frac{n}{2} \\
= & -\frac{n}{2}\ \left[\log(2\pi) + \ \log(\hat{\varepsilon}'\hat{\varepsilon}/n) + 1\right]
\end{aligned} \tag{a.3.10.35}$$

Noting then that $|\theta| = k+1$ (corresponding to k coefficients for β and one for σ), AIC obtains as:

$$\begin{aligned}
& AIC = 2|\theta|/n - 2\ \hat{L}/n \\
= & \ 2(k+1)/n - 2\left(-\frac{n}{2}\ \left[\log(2\pi) + \ \log(\hat{\varepsilon}'\hat{\varepsilon}/n) + 1\right]\right)/n \\
= & \ 2\ (k+1)/n\ +\ \log(2\pi)\ +\ \log(\hat{\varepsilon}'\hat{\varepsilon}/n) + 1
\end{aligned} \tag{a.3.10.36}$$

Using similar calculations, BIC immediately obtains as:

$$BIC = (k+1)\ \log(n)/n\ +\ \log(2\pi)\ +\ \log(\hat{\varepsilon}'\hat{\varepsilon}/n) + 1 \tag{a.3.10.37}$$

derivations for equation (3.10.19)

Rearrange:

$$
\begin{aligned}
& f(y) \\
= & \int (2\pi\sigma)^{-n/2} \exp\left(-\frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma}\right) \times (2\pi)^{-k/2}|V|^{-1/2} \exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right) d\beta \\
= & \int (2\pi)^{-(n+k)/2}\ \sigma^{-n/2}\ |V|^{-1/2} \times \exp\left(-\frac{1}{2}\left[(y-X\beta)'\sigma^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)\right]\right) d\beta
\end{aligned}
\tag{a.3.10.38}
$$

Consider the term square brackets:

$$
\begin{aligned}
& (y-X\beta)'\sigma^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b) \\
= & y'\sigma^{-1}y+\beta'X'\sigma^{-1}X\beta-2\beta'X'\sigma^{-1}y+\beta'V^{-1}\beta+b'V^{-1}b-2\beta'V^{-1}b \\
= & \beta'(V^{-1}+\sigma^{-1}X'X)\beta-2\beta'(V^{-1}b+\sigma^{-1}X'y)+y'\sigma^{-1}y+b'V^{-1}b \\
= & \beta'(V^{-1}+\sigma^{-1}X'X)\beta-2\beta'\bar{V}^{-1}\bar{V}(V^{-1}b+\sigma^{-1}X'y)+\bar{b}\bar{V}^{-1}\bar{b}+y'\sigma^{-1}y+b'V^{-1}b-\bar{b}\bar{V}^{-1}\bar{b}
\end{aligned}
\tag{a.3.10.39}
$$

Define:

$$
\bar{V}=(V^{-1}+\sigma^{-1}X'X)^{-1} \qquad \bar{b}=\bar{V}(V^{-1}b+\sigma^{-1}X'y)
\tag{a.3.10.40}
$$

Then (a.3.10.39) reformulates:

$$
\begin{aligned}
= & \beta'\bar{V}^{-1}\beta-2\beta'\bar{V}^{-1}\bar{b}+\bar{b}\bar{V}^{-1}\bar{b}+y'\sigma^{-1}y+b'V^{-1}b-\bar{b}\bar{V}^{-1}\bar{b} \\
= & (\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+y'\sigma^{-1}y+b'V^{-1}b-\bar{b}\bar{V}^{-1}\bar{b}
\end{aligned}
\tag{a.3.10.41}
$$

Substitute back in (a.3.10.38):

$$
\begin{aligned}
& f(y) \\
= & \int (2\pi)^{-(n+k)/2}\ \sigma^{-n/2}\ |V|^{-1/2} \times \exp\left(-\frac{1}{2}\left[(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+y'\sigma^{-1}y+b'V^{-1}b-\bar{b}\bar{V}^{-1}\bar{b}\right]\right) d\beta \\
= & (2\pi)^{-(n+k)/2}\ \sigma^{-n/2}\ |V|^{-1/2}(2\pi)^{k/2}|\bar{V}|^{1/2} \times \exp\left(-\frac{1}{2}\left[y'\sigma^{-1}y+b'V^{-1}b-\bar{b}\bar{V}^{-1}\bar{b}\right]\right) \\
\times & \int (2\pi)^{-k/2}|\bar{V}|^{-1/2} \exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right) d\beta \\
= & (2\pi)^{-n/2}\ \sigma^{-n/2}|\bar{V}|^{1/2}|V|^{-1/2} \times \exp\left(-\frac{1}{2}\left[y'\sigma^{-1}y+b'V^{-1}b-\bar{b}\bar{V}^{-1}\bar{b}\right]\right) \\
\times & \int (2\pi)^{-k/2}|\bar{V}|^{-1/2} \exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right) d\beta
\end{aligned}
\tag{a.3.10.42}
$$

derivations for equation (3.10.21)

Consider the term:

$$\begin{aligned}
& (2\pi)^{-n/2}\ \sigma^{-n/2}|\bar{V}|^{1/2}|V|^{-1/2} \\
=\ & (2\pi)^{-n/2}\ \sigma^{-n/2}|(V^{-1}+\sigma^{-1}X'X)^{-1}|^{1/2}|V|^{-1/2} \\
=\ & (2\pi)^{-n/2}\ \sigma^{-n/2}|V^{-1}+\sigma^{-1}X'X|^{-1/2}|V|^{-1/2} \\
=\ & (2\pi)^{-n/2}\ \sigma^{-n/2}(|V||V^{-1}+\sigma^{-1}X'X|)^{-1/2} \\
=\ & (2\pi)^{-n/2}\ \sigma^{-n/2}|I_k+\sigma^{-1}VX'X|^{-1/2} \qquad \text{(a.3.10.43)}
\end{aligned}$$

Hence:

$$f(y) = (2\pi)^{-n/2}\ \sigma^{-n/2}|I_k+\sigma^{-1}VX'X|^{-1/2}\exp\left(-\frac{1}{2}\left[y'\sigma^{-1}y+b'V^{-1}b-\bar{b}\bar{V}^{-1}\bar{b}\right]\right) \qquad \text{(a.3.10.44)}$$

derivations for equation (3.10.23)

Rearrange the expression:

$$\begin{aligned}
& f(y) \\
=\ & \int\int (2\pi\sigma)^{-n/2}\ \exp\left(-\frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma}\right) \\
\times\ & (2\pi)^{-k/2}|\sigma V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'(\sigma V)^{-1}(\beta-b)\right)\times\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)d\beta d\sigma \\
=\ & \int\int (2\pi\sigma)^{-n/2}\ (2\pi)^{-k/2}|\sigma V|^{-1/2}\ \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1} \\
\times\ & \exp\left(-\frac{1}{2\sigma}\left[(y-X\beta)'(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)+\delta\right]\right)d\beta d\sigma \qquad \text{(a.3.10.45)}
\end{aligned}$$

Consider the term in square brackets and complete the squares:

$$\begin{aligned}
& (y-X\beta)'(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)+\delta \\
=\ & y'y+\beta'X'X\beta-2\beta'X'y+\beta'V^{-1}\beta+b'V^{-1}b-2\beta'V^{-1}b+\delta \\
=\ & \beta'(V^{-1}+X'X)\beta-2\beta'(V^{-1}b+X'y)+\delta+y'y+b'V^{-1}b \\
=\ & \beta'(V^{-1}+X'X)\beta-2\beta'\bar{V}^{-1}\bar{V}(V^{-1}b+X'y)+\delta+y'y+b'V^{-1}b+\bar{b}'\bar{V}^{-1}\bar{b}-\bar{b}'\bar{V}^{-1}\bar{b} \qquad \text{(a.3.10.46)}
\end{aligned}$$

Define:

$$\bar{V}=(V^{-1}+X'X)^{-1} \qquad \bar{b}=\bar{V}(V^{-1}b+X'y) \qquad \bar{\delta}=\delta+y'y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b} \qquad \text{(a.3.10.47)}$$

Then (a.3.10.46) rewrites:

$$\begin{aligned}
=\ & \beta'\bar{V}^{-1}\beta-2\beta'\bar{V}^{-1}\bar{b}+\bar{b}'\bar{V}^{-1}\bar{b}+\bar{\delta} \\
=\ & (\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+\bar{\delta} \qquad \text{(a.3.10.48)}
\end{aligned}$$

Substituting back in (a.3.10.45):

$$
\begin{aligned}
& f(y) \\
= \; & \int\int (2\pi\sigma)^{-n/2} \; (2\pi)^{-k/2} |\sigma V|^{-1/2} \; \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \sigma^{-\alpha/2-1} \\
\times \; & \exp\left(-\frac{1}{2\sigma} \left[(\beta-\bar{b})' \bar{V}^{-1} (\beta-\bar{b}) + \bar{\delta} \right] \right) d\beta d\sigma \\
= \; & \int\int (2\pi)^{-n/2} \; (2\pi)^{-k/2} |\sigma V|^{-1/2} \; \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \\
\times \; & \sigma^{-(\alpha+n)/2-1} \exp\left(-\frac{1}{2\sigma} \left[(\beta-\bar{b})' \bar{V}^{-1} (\beta-\bar{b}) + \bar{\delta} \right] \right) d\beta d\sigma
\end{aligned} \tag{a.3.10.49}
$$

define:

$$
\bar{\alpha} = \alpha + n \tag{a.3.10.50}
$$

Then (a.3.10.49) rewrites:

$$
\begin{aligned}
= \; & \int\int (2\pi)^{-n/2} \; (2\pi)^{-k/2} |\sigma V|^{-1/2} \; \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \\
\times \; & \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{1}{2\sigma} \left[(\beta-\bar{b})' \bar{V}^{-1} (\beta-\bar{b}) + \bar{\delta} \right] \right) d\beta d\sigma \\
= \; & (2\pi)^{-n/2} \; |\sigma V|^{-1/2} |\sigma \bar{V}|^{1/2} \; \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \; \frac{\Gamma(\bar{\alpha}/2)}{\bar{\delta}/2^{\bar{\alpha}/2}} \\
\times \; & \int\int (2\pi)^{-k/2} |\sigma \bar{V}|^{-1/2} \exp\left(-\frac{1}{2} (\beta-\bar{b})' (\sigma \bar{V})^{-1} (\beta-\bar{b}) \right) \times \frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)} \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma} \right) d\beta d\sigma \\
= \; & 2^{-n/2} \pi^{-n/2} \; |V|^{-1/2} |\bar{V}|^{1/2} \; \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \; \frac{2^{(\alpha+n)/2}}{2^{\alpha/2}} \; \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
\times \; & \int\int (2\pi)^{-k/2} |\sigma \bar{V}|^{-1/2} \exp\left(-\frac{1}{2} (\beta-\bar{b})' (\sigma \bar{V})^{-1} (\beta-\bar{b}) \right) \times \frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)} \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma} \right) d\beta d\sigma \\
= \; & \pi^{-n/2} \; |V|^{-1/2} |\bar{V}|^{1/2} \; \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \; \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
\times \; & \int\int (2\pi)^{-k/2} |\sigma \bar{V}|^{-1/2} \exp\left(-\frac{1}{2} (\beta-\bar{b})' (\sigma \bar{V})^{-1} (\beta-\bar{b}) \right) \times \frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)} \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma} \right) d\beta d\sigma
\end{aligned} \tag{a.3.10.51}
$$

derivations for equation (3.10.25)

Reformulate the expression:

$$
\begin{aligned}
& \pi^{-n/2}\, |V|^{-1/2} |\bar{V}|^{1/2}\, \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\, \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
= \;& \pi^{-n/2}\, |V|^{-1/2} |(V^{-1}+X'X)^{-1}|^{1/2}\, \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\, \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
= \;& \pi^{-n/2}\, |V|^{-1/2} |(V^{-1}+X'X)|^{-1/2}\, \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\, \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
= \;& \pi^{-n/2}\, |V(V^{-1}+X'X)|^{-1/2}\, \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\, \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
= \;& \pi^{-n/2}\, |I_k + VX'X|^{-1/2}\, \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\, \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \qquad \text{(a.3.10.52)}
\end{aligned}
$$

derivations for equation (3.10.27)

Rearrange the expression:

$$
\begin{aligned}
& f(y) \\
\approx\; & \frac{f(y|\beta^*,\sigma^*)\pi(\beta^*,\sigma^*)}{\pi(\sigma^*|y,\beta^*)\times\frac{1}{J}\sum_{j=1}^{J}\pi(\beta^*|\sigma^{(j)},y)} \\
=\; & (2\pi\sigma)^{-n/2}\ \exp\left(-\frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma}\right)\times\frac{(2\pi)^{-k/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi)^{-k/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)} \\
\times\; & \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\sigma^{-\bar{\alpha}/2-1}\ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
=\; & 2^{-n/2}\pi^{-n/2}\sigma^{-n/2}\ \exp\left(-\frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma}\right)\times\frac{(2\pi)^{-k/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi)^{-k/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)} \\
\times\; & \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{2^{(\alpha+n)/2}}{2^{\alpha/2}}\frac{\delta^{\alpha/2}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)}{\bar{\delta}^{\bar{\alpha}/2}\ \sigma^{-(\alpha+n)/2-1}\ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
=\; & 2^{-n/2}\pi^{-n/2}\sigma^{-n/2}\ \times\frac{(2\pi)^{-k/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi)^{-k/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)} \\
\times\; & \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{2^{(\alpha+n)/2}}{2^{\alpha/2}}\frac{\delta^{\alpha/2}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta+(y-X\beta)'(y-X\beta)}{2\sigma}\right)}{\bar{\delta}^{\bar{\alpha}/2}\ \sigma^{-(\alpha+n)/2-1}\ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
=\; & 2^{-n/2}\pi^{-n/2}\sigma^{-n/2}\ \times\frac{(2\pi)^{-k/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi)^{-k/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)} \\
\times\; & \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{2^{(\alpha+n)/2}}{2^{\alpha/2}}\frac{\delta^{\alpha/2}\sigma^{-\alpha/2-1}\exp\left(-\frac{\bar{\delta}}{2\sigma}\right)}{\bar{\delta}^{\bar{\alpha}/2}\ \sigma^{-(\alpha+n)/2-1}\ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
=\; & \pi^{-n/2}\frac{|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \\
=\; & \pi^{-n/2}\frac{\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}|V|^{1/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \\
=\; & \pi^{-n/2}\frac{\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}|V|^{1/2}|(V^{-1}+\sigma^{-1}X'X)^{-1}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \\
=\; & \pi^{-n/2}\frac{\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}|V|^{1/2}|V^{-1}+\sigma^{-1}X'X|^{1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \\
=\; & \pi^{-n/2}\frac{\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}|I_k+\sigma^{-1}VX'X|^{1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \qquad \text{(a.3.10.53)}
\end{aligned}
$$

derivations for equation (3.10.29)

Substitute for the functions and rearrange:

$$
\begin{aligned}
&\frac{1}{f(y)} \\
\approx\ & \frac{1}{J}\sum_{j=1}^{J} \frac{g(\theta^{(j)})}{f(y|\beta^{(j)},\sigma^{(j)},\gamma^{(j)})\ \pi(\beta^{(j)})\ \pi(\sigma^{(j)})\ \pi(\gamma^{(j)})} \\
=\ & \frac{1}{J}\sum_{j=1}^{J} \frac{\omega^{-1}(2\pi)^{-(k+h+1)/2}|\hat{\Sigma}|^{-1/2}\exp\left(-\frac{1}{2}(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\right)\mathbb{1}(\theta\in\hat{\Theta})}{\left[\begin{array}{l}(2\pi\sigma)^{-n/2}\ |W|^{-1/2}\ \exp\left(-\frac{1}{2}\frac{(y-X\beta)'W^{-1}(y-X\beta)}{\sigma}\right)\times(2\pi)^{-k/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right) \\ \times\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)\times(2\pi)^{-h/2}|Q|^{-1/2}\exp\left(-\frac{1}{2}(\gamma-g)'Q^{-1}(\gamma-g)\right)\end{array}\right]} \\
=\ & \frac{1}{J}\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\ \ \omega^{-1}(2\pi)^{(n+k+h-(k+h+1))/2}\ |\hat{\Sigma}|^{-1/2}\ |W|^{1/2}\ |V|^{1/2}\ |Q|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\ \ \sigma^{(\alpha+n)/2+1} \\
&\times\exp\left(\frac{1}{2}\left[\begin{array}{l}(y-X\beta)'(\sigma W)^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)+\delta\sigma^{-1} \\ +(\gamma-g)'Q^{-1}(\gamma-g)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right) \\
=\ & (\omega J)^{-1}(2\pi)^{(n-1)/2}\ |\hat{\Sigma}|^{-1/2}\ |V|^{1/2}\ |Q|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}} \\
&\times\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\ |W|^{1/2}\ \sigma^{(\alpha+n)/2+1}\ \exp\left(\frac{1}{2}\left[\begin{array}{l}(y-X\beta)'(\sigma W)^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b) \\ +\delta\sigma^{-1}+(\gamma-g)'Q^{-1}(\gamma-g)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right)
\end{aligned}
\tag{a.3.10.54}
$$

Using logs on both sides yields:

$$
\begin{aligned}
&-log(f(y))\approx log\left((\omega J)^{-1}(2\pi)^{(n-1)/2}\ |\hat{\Sigma}|^{-1/2}\ |V|^{1/2}\ |Q|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\right) \\
&+log\left(\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\ |W|^{1/2}\ \sigma^{(\alpha+n)/2+1}\exp\left(\frac{1}{2}\left[\begin{array}{l}(y-X\beta)'(\sigma W)^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b) \\ +\delta\sigma^{-1}+(\gamma-g)'Q^{-1}(\gamma-g)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right)\right)
\end{aligned}
\tag{a.3.10.55}
$$

or:

$$
\begin{aligned}
&log(f(y))\approx -log\left((\omega J)^{-1}(2\pi)^{(n-1)/2}\ |\hat{\Sigma}|^{-1/2}\ |V|^{1/2}\ |Q|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\right) \\
&-log\left(\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\ |W|^{1/2}\ \sigma^{(\alpha+n)/2+1}\exp\left(\frac{1}{2}\left[\begin{array}{l}(y-X\beta)'(\sigma W)^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b) \\ +\delta\sigma^{-1}+(\gamma-g)'Q^{-1}(\gamma-g)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right)\right)
\end{aligned}
\tag{a.3.10.56}
$$

derivations for equation (3.10.32)

Substitute for the functions and rearrange:

$$\begin{aligned}
&\frac{1}{f(y)} \\
\approx\ & \frac{1}{J}\sum_{j=1}^{J}\frac{g(\theta^{(j)})}{f(y|\beta^{(j)},\sigma^{(j)},\phi^{(j)})\ \pi(\beta^{(j)})\ \pi(\sigma^{(j)})\ \pi(\phi^{(j)})} \\
=\ & \frac{1}{J}\sum_{j=1}^{J}\frac{\omega^{-1}(2\pi)^{-(k+q+1)/2}|\hat{\Sigma}|^{-1/2}\exp\left(-\frac{1}{2}(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\right)\mathbb{1}(\theta\in\hat{\Theta})}{\left[\begin{array}{l}(2\pi\sigma)^{-T/2}\ \exp\left(-\frac{1}{2}(\varepsilon-E\phi)'\sigma^{-1}(\varepsilon-E\phi)\right)\times(2\pi)^{-k/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)\\ \times\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)\times(2\pi)^{-q/2}|Z|^{-1/2}\exp\left(-\frac{1}{2}(\phi-p)'Z^{-1}(\phi-p)\right)\end{array}\right]} \\
=\ & \frac{1}{J}\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\ \ \omega^{-1}(2\pi)^{(T+k+q-(k+q+1))/2}\ |\hat{\Sigma}|^{-1/2}\ |V|^{1/2}\ |Z|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\ \ \sigma^{(\alpha+T)/2+1} \\
&\times\exp\left(\frac{1}{2}\left[\begin{array}{l}(\varepsilon-E\phi)'\sigma^{-1}(\varepsilon-E\phi)+(\beta-b)'V^{-1}(\beta-b)\\ +\delta\sigma^{-1}+(\phi-p)'Z^{-1}(\phi-p)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right) \\
=\ & (\omega J)^{-1}(2\pi)^{(T-1)/2}\ |\hat{\Sigma}|^{-1/2}\ |V|^{1/2}\ |Z|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}} \\
&\times\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\sigma^{(\alpha+T)/2+1}\exp\left(\frac{1}{2}\left[\begin{array}{l}(\varepsilon-E\phi)'\sigma^{-1}(\varepsilon-E\phi)+(\beta-b)'V^{-1}(\beta-b)\\ +\delta\sigma^{-1}+(\phi-p)'Z^{-1}(\phi-p)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right)
\end{aligned} \tag{a.3.10.57}$$

Using logs on both sides yields:

$$\begin{aligned}
&-log(f(y))\approx log\left((\omega J)^{-1}(2\pi)^{(T-1)/2}\ |\hat{\Sigma}|^{-1/2}\ |V|^{1/2}\ |Z|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\right) \\
&+log\left(\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\sigma^{(\alpha+T)/2+1}\exp\left(\frac{1}{2}\left[\begin{array}{l}(\varepsilon-E\phi)'\sigma^{-1}(\varepsilon-E\phi)+(\beta-b)'V^{-1}(\beta-b)\\ +\delta\sigma^{-1}+(\phi-p)'Z^{-1}(\phi-p)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right)\right)
\end{aligned} \tag{a.3.10.58}$$

or:

$$\begin{aligned}
&log(f(y))\approx -log\left((\omega J)^{-1}(2\pi)^{(T-1)/2}\ |\hat{\Sigma}|^{-1/2}\ |V|^{1/2}\ |Z|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\right) \\
&-log\left(\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\sigma^{(\alpha+T)/2+1}\exp\left(\frac{1}{2}\left[\begin{array}{l}(\varepsilon-E\phi)'\sigma^{-1}(\varepsilon-E\phi)+(\beta-b)'V^{-1}(\beta-b)\\ +\delta\sigma^{-1}+(\phi-p)'Z^{-1}(\phi-p)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right)\right)
\end{aligned} \tag{a.3.10.59}$$

Bibliography